SOLVED PROBLEMS IN INTEGRAL CALCULUS

Jonathan B. Cabero
Lorina G. Salamat
Antonina C. Sta. Maria

សុខ ពិសិដ្ឋ

Table of Contents

OPICS :

Indefinite Integral

$$\int f(x)dx = F(x)+C \text{ if } d[F(x)+C] = f(x)$$

Fundamental Integration Formulas

$$\int du = u + C$$

$$\int u^n du = \frac{u^{n+1}}{n+1} + C$$

$$\int au du = a\int u du$$

$$\int (f \pm g) du = \int f du \pm \int g du$$

Examples

1. $\int x^3 dx = \frac{x^{3+1}}{3+1} = \frac{x^4}{4} + C$

2. $\int x^{3/2} dx = \frac{x^{3/2+1}}{3/2+1} + C = \frac{x^{5/2}}{5/2} + C = \frac{2}{5}x^{5/2} + C$

3. $\int \frac{dx}{\sqrt{x}} = \int x^{-1/2} dx = \frac{x^{-1/2+1}}{1/2+1} + C = \frac{x^{1/2}}{1/2} + C = 2x^{1/2} + C$

4. $\int \frac{dy}{y^6} = \int y^{-6} dy = \frac{y^{-6+1}}{-6+1} + C = \frac{y^{-5}}{-5} + C = \frac{-1}{5y^5} + C$

5. $\int (2x+3)^2 dx = \int (2x^2 + 12x + 9) dx = 2\int x^2 dx + 12\int x dx + 9\int dx = 2\frac{x^3}{3} + 12\frac{x^2}{2} + 9x + C$

   $= \frac{2}{3}x^3 + 6x^2 + 9x + C$

6. $\int \frac{(4x^2 - 5x + 6)}{x^2} dx = \int \left(4 - \frac{5}{x} + \frac{6}{x^2}\right) dx = 4\int dx - 5\int \frac{dx}{x} + 6\int \frac{dx}{x^2} = 4x - 5\ln x - \frac{6}{x} + C$

7. $\int \frac{(1-\sqrt{x})^2}{x} dx = \int \frac{1-2\sqrt{x}+x}{x} dx = \int \left(\frac{1}{x} - 2x^{-1/2} + 1\right) dx = \ln x - \frac{2x^{1/2}}{\frac{1}{2}} + x + C$

$= \ln x - 4x^{1/2} + x + C$

8. $\int \sqrt{x}(x^2 - 3x + 2) dx = \int x^{1/2}(x^2 - 3x + 2)) dx = \int (x^{5/2} - 3x^{3/2} + 2x^{1/2}) dx$

$= \frac{x^{7/2}}{\frac{7}{2}} - \frac{3x^{5/2}}{\frac{5}{2}} + \frac{2x^{3/2}}{\frac{3}{2}} + C = \frac{2}{7}x^{7/2} - \frac{6}{5}x^{5/2} + \frac{4}{3}x^{3/2} + C$

9. $\int \frac{(t-1)^2}{\sqrt{t}} dt = \int \frac{(t^2 - 2t + 1)}{t^{1/2}} dt = \int (t^{3/2} - 2t^{1/2} + t^{-1/2}) dt = \frac{t^{5/2}}{\frac{5}{2}} - \frac{2t^{3/2}}{\frac{3}{2}} + \frac{t^{1/2}}{\frac{1}{2}} + C$

$= \frac{2}{5}t^{5/2} - \frac{4}{3}t^{3/2} + 2t^{1/2} + C$

10. $\int \left(t + \frac{1}{t}\right)^3 dt = \int \left(t^3 + 3t + 3\frac{1}{t} + \frac{1}{t^3}\right) dt = \frac{t^4}{4} + \frac{3t^2}{2} + 3\ln t - \frac{1}{2t^2} + C$

11. $\int (8x^3 - 6x^2 + 12x - 9)\, dx$

Solution.

$\int (8x^3 - 6x^2 + 12x - 9)\, dx$
$= 8\int x^3\, dx - 6 \int x^2 + 12 \int x\, dx - 9 \int dx$

$= \frac{8x^4}{4} - \frac{6x^3}{3} + \frac{12x^2}{2} - 9x + C$

$= 2x^4 - 2x^3 + 6x^2 - 9x + C$

12. $\int \frac{(x^2 - 1)^3}{\sqrt{x}} dx$

Solution.

$\int \frac{(x^6 - 3x^4 + 3x^2 - 1)}{x^{1/2}} dx$

$= \int \frac{x^6}{x^{1/2}} dx - 3 \int \frac{x^4}{x^{1/2}} dx + 3\int \frac{x^2}{x^{1/2}} dx - \int \frac{dx}{x^{1/2}}$

$= \int x^{11/2}\, dx - 3 \int x^{7/2}\, dx + 3\int x^{3/2}\, dx - \int x^{-½}\, dx$

$$= \frac{x^{13/2}}{13/2} - \frac{3x^{9/2}}{9/2} + \frac{3x^{5/2}}{5/2} + \frac{x^{1/2}}{1/2} + C$$

$$= \frac{2x^{13/2}}{13} - \frac{2x^{9/2}}{3} + \frac{6x^{5/2}}{5} + 2x^{1/2} + C$$

13. $\int \frac{1}{3}\left(e^{3x} - e^{-3x}\right)^2 dx$

$$= \frac{1}{3}\int\left(e^{6x} - 2 + e^{-6x}\right)dx$$

$$= \frac{1}{3}\left[\int e^{6x}dx - 2\int dx + \int e^{-6x}dx\right]$$

$$= \frac{1}{3}\left[\frac{1}{6}e^{6x} - 2x - \frac{1}{6}e^{-6x}\right] + c$$

$$= \frac{1}{18}\left(e^{6x} - 12x - e^{-6x}\right) + c$$

14. $\int(5\cos 4t - 3\sin 2t)dt$

$$= 5\int\cos 4t dt - 3\int\sin 2t dt$$

$$= \frac{5}{4}\sin 4t + \frac{3}{2}\cos 2t + C$$

15. $\int\left(\sec^2 4t + \csc^2 5t\right)dt$

$$= \frac{1}{4}\tan 4t - \frac{1}{5}\cot 5t + C$$

Chain Rule for Integration

$$\int f[g(x)]g'(x)dx = F(g(x)) + C$$

Examples

1. $\int(3x-1)^4 dx = \int u^4 \cdot \frac{du}{3} = \frac{1}{3}\int u^4 du = \frac{1}{3}\cdot\frac{u^5}{5} + C = \frac{1}{15}u^5 + C = \frac{1}{15}(3x-1)^5 + C$

let u = 3x – 1, $du = 3dx$ or $dx = \dfrac{du}{3}$

2. $\int (2x^3+5)^{10} x^2 dx = \int u^{10} \cdot \dfrac{du}{6} = \dfrac{1}{6}\int u^{10} du = \dfrac{1}{6}\cdot\dfrac{u^{11}}{11} + C = \dfrac{1}{66}u^{11} + C = \dfrac{1}{66}(2x^3+5)^{11} + C$

let u = $2x^3$ +5, $du = 6x^2dx$ or $x^2dx = \dfrac{du}{6}$

3.

$$\int \sqrt{4-x^2}\, xdx = \int u^{1/2}\cdot\left(\frac{-du}{2}\right) = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{u^{3/2}}{\frac{3}{2}} + C = -\frac{1}{3}u^{3/2} + C = -\frac{1}{3}(4-x^2)^{3/2} +$$

let u = 4 – x^2, $du = -2xdx$ or $\dfrac{-du}{2} = xdx$

4. $\int \dfrac{xdx}{4x^2-5} = \int \dfrac{\frac{du}{8}}{u} = \dfrac{1}{8}\int\dfrac{du}{u} = \dfrac{1}{8}\ln|u| + C = \dfrac{1}{8}\ln|4x^2-5| + C$

let u = $4x^2$ – 5, $du = 8xdx$ or $xdx = \dfrac{du}{8}$

5.

$$\int \frac{x^2dx}{\sqrt{x^3-1}} = \int \frac{\frac{du}{3}}{u^{1/2}} = \frac{1}{3}\int u^{-1/2}du = \frac{1}{3}\frac{u^{1/2}}{\frac{1}{2}} + C = \frac{2}{3}u^{1/2} + C = \frac{2}{3}(x^3-1)^{1/2} + C = \frac{2}{3}\sqrt{x^3-1} + C$$

let u = x^3 – 1, $du = 3x^2dx$ or $x^2dx = \dfrac{du}{3}$

6. $\int \dfrac{xdx}{(5x^2+3)^4} = \int \dfrac{\frac{du}{10}}{u^4} = \dfrac{1}{10}\int u^{-4}du = \dfrac{1}{10}\cdot\dfrac{u^{-3}}{-3} + C = -\dfrac{1}{30u^3} + C = \dfrac{1}{30(5x^2+3)^3} + C$

let u =$5x^2$ +3, $du = 10xdx$ or $xdx = \dfrac{du}{10}$

7. $\int (4x+5)^3 dx$

Let *u = 4x + 5, du = 4 dx* or $\dfrac{1}{4}du = dx$

$\int (4x+5)^3 dx = \int u^3 \cdot \dfrac{1}{4}du$

$$= \frac{1}{4}\int u^3 \cdot du = \frac{1}{4}\cdot\frac{u^4}{4} + C = \frac{u^4}{16} + C$$, replacing back u by 4x + 5

$$= \frac{1}{16}(4x+5)^4 + C$$

8. $\int x\sqrt{4x^2+1}dx$

Let $u = 4x^2 + 1$, $du = 8xdx$ *or* $\frac{1}{8}du = xdx$

$$\int x\sqrt{4x^2+1}dx = \int x\sqrt{4x^2+1}dx$$

$$= \int \sqrt{4x^2+1}xdx$$

$$= \int u^{1/2} \cdot \frac{1}{8}du$$

$$= \frac{1}{8}\int u^{1/2}du$$

$$= \frac{1}{8}\frac{u^{3/2}}{\frac{3}{2}} + C$$

$$= \frac{1}{12}u^{3/2} + C$$

replacing back u by $4x^2 + 1$

$$= \frac{1}{12}\left(4x^2+1\right)^{3/2} + C \text{ or } \frac{1}{12}\sqrt{(4x^2+1)^3} + C$$

9. $\int \frac{6x^2dx}{\left(x^3+1\right)^5}$

Let $u = x^3 + 1$, $du = 3x^2$ dx or $\frac{1}{3}du = x^2dx$

$$\int \frac{6x^2dx}{\left(x^3+1\right)^5} = 6\int \left(x^3+1\right)^{-5} \cdot x^2dx$$

$$= 6\int u^{-5} \cdot \frac{1}{3} du$$

$$= 2\int u^{-5} du$$

$$= 2\frac{u^{-4}}{-4} + C$$

$$= \frac{1}{-2u^4} + C$$

replacing back u by $x^3 + 1$

$$= \frac{-1}{2(x^3+1)^4} + C$$

10. $\int (4x^3+9)^2 x^5 dx$

Let $u = 4x^3 + 9$, $du = 12x^2\, dx$ or $\frac{1}{12} du = x^2 dx$

$$\int (4x^3+9)^2 x^5 dx = \int (4x^3+9)^2 \cdot x^3 \cdot x^2 dx$$

$u = 4x^3 + 1$, $u - 1 = 4x^3$ or $\frac{u-1}{4} = x^3$

$$\int (4x^3+9)^2 \cdot x^3 \cdot x^2 dx = \int u^2 \cdot \frac{u-1}{4} \cdot \frac{1}{12} du$$

$$= \frac{1}{48} \int u^2 (u-1) du$$

$$= \frac{1}{48} \int (u^3 - u^2) du$$

$$= \frac{1}{48} [\int u^3 du - \int u^2 du]$$

$$= \frac{1}{48} \left[\frac{u^4}{4} - \frac{u^3}{3}\right] + C$$

$$= \frac{1}{48} u^3 \left(\frac{3u-4}{12}\right) + C$$

replacing u by $4x^3 + 9$

$$= \frac{1}{576} (4x^3+1)^3 [3(4x^3+1) - 4] + C$$

$$= \frac{1}{576} (4x^3+1)^3 (12x^3 - 1) + C$$

Integral of Exponential Functions

$$\int a^u\,du = \frac{a^u}{\ln a} + C \qquad \int a^{bx}\,dx = \frac{a^{bx}}{b\ln a} + C$$

$$\int e^u\,du = e^u + C \qquad \int e^{bx}\,dx = \frac{1}{b}e^{bx} + C$$

Examples

1. $\int 4^x\,dx = \dfrac{4^x}{\ln 2} + C$

2. $\int 3^{2x}\,dx = \dfrac{1}{2}\int 3^u\,du = \dfrac{1}{2}\cdot\dfrac{3^u}{\ln 3} + C = \dfrac{3^{2x}}{2\ln 3} + C$

   let u =2x, du = 2dx or $dx = \dfrac{du}{2}$

3. $\int 2^{2x^2+1}\,x dx = \dfrac{1}{4}\int 2^u\,du = \dfrac{1}{4}\cdot\dfrac{2^u}{\ln 2} + C = \dfrac{2^{2x^2+1}}{4\ln 2} + C$

   let u =$2x^2$ + 1, du = 4xdx or $xdx = \dfrac{du}{4}$

4. $\int e^x\,dx = e^x + C$

5. $\int e^{3x}\,dx = \dfrac{1}{3}\int e^u\,du = \dfrac{1}{3}e^{3x} + C$

   let u = 3x, du =3dx or $dx = \dfrac{du}{3}$

6. $\int \dfrac{\left(e^{2x} + e^{-2x}\right)}{2}dx = \dfrac{1}{2}\int\left(e^{2x} + e^{-2x}\right)dx = \dfrac{1}{2}\left(\dfrac{e^{2x}}{2} + \dfrac{e^{-2x}}{-2}\right) + C = \dfrac{1}{4}\left(e^{2x} - e^{-2x}\right) + C$

7. $\int\left(e^{4v} - \dfrac{4}{v}\right)dv = \dfrac{1}{4}e^{4v} - 4\ln v + C$

8. $\int\left(e^{2r} - e^{-2r}\right)^2 dr = \int\left(e^{4r} - 2 + e^{-4r}\right)dr = \dfrac{1}{4}e^{4r} - 2r - \dfrac{1}{4}e^{-4r} + C$

9. $\int\left(e^{2t} - \dfrac{1}{e^{3t}} + \dfrac{1}{t}\right)dt = \int\left(e^{2t} - e^{-3t} + \dfrac{1}{t}\right)dt = \dfrac{1}{2}e^{2t} + \dfrac{1}{3}e^{-3t} + \ln t + C$

10. $\int\left(e^{x}+\frac{1}{e^{x}}\right)^{2}dx=\int\left(e^{2x}+2+\frac{1}{e^{2x}}\right)dx=\int\left(e^{2x}+2+e^{-2x}\right)dx=\frac{1}{2}e^{2x}+2x-\frac{1}{2}e^{-2x}+C$

11. $\int e^{2x}\left(1+e^{2x}\right)^{3}dx=\frac{1}{2}\int u^{3}du=\frac{1}{2}\cdot\frac{1}{4}u^{4}+C=\frac{1}{8}\left(1+e^{2x}\right)^{4}+C$

let u = 1 +e^{2x}, du =2e^{2x}dx, $e^{2x}dx=\frac{du}{2}$

12. $\int ze^{4z^{2}}dz=\frac{1}{8}\int e^{u}du=\frac{1}{8}e^{u}+C=\frac{1}{8}e^{4z^{2}}+C$

let $u=4z^{2},du=8zdz,zdz=\frac{du}{8}$

13. $\int\frac{e^{x}dx}{\left(e^{x}+1\right)^{3/2}}=\int\left(e^{x}+1\right)^{-3/2}e^{x}dx=\int u^{-3/2}du=\frac{u^{-1/2}}{-\frac{1}{2}}+C=-2\left(e^{x}+1\right)^{-1/2}+C$

let u = e^x + 1, du = e^xdx

14.

$\int e^{3y}\left(1-e^{3y}\right)^{2/3}dy=-\frac{1}{3}\int u^{2/3}du=-\frac{1}{3}\frac{u^{5/3}}{\frac{5}{3}}+C=-\frac{1}{5}u^{5/3}+C=-\frac{1}{5}\left(1-e^{3y}\right)+C$

let u = 1 – e^{ey} , du =-3e^{3y}dy, e^{3y}dy=$-\frac{du}{3}$

15. $\int\frac{\left(e^{x}+2\right)^{2}}{e^{x}}dx\int\frac{\left(e^{2x}+4e^{x}+4\right)}{e^{x}}dx=\int\left(e^{x}+4+4e^{-x}\right)dx=e^{x}+4x-4e^{-x}+C$

16. $\int\frac{e^{3t}dt}{\left(1+4e^{3t}\right)^{2}}=\frac{1}{12}\int u^{-2}du=-\frac{1}{12}\cdot\frac{u^{-1}}{-1}+C=-\frac{1}{12}\left(1+4e^{3t}\right)^{-1}+C$

let u = 1+ 4e^{3t}, du = 12e^{3t}dt, $e^{3t}dt=\frac{du}{12}$

17. $\int\frac{e^{2x}dx}{4-3e^{2x}}=-\frac{1}{6}\int\frac{du}{u}=-\frac{1}{6}\ln u+C=-\frac{1}{6}\ln\left|4-3e^{2x}\right|+C$

let u = 4 – 3e^{2x}, $du=-6e^{2x}dx,e^{2x}dx=-\frac{du}{6}$

18. $\int\frac{e^{\tan 3\theta}d\theta}{\cos^{2}3\theta}=\int e^{\tan 3\theta}\sec^{2}3\theta d\theta=\frac{1}{3}e^{\tan 3\theta}+C$

let u = $\tan 3\theta, du = 3\sec^2 3\theta, \sec^2 3\theta = \dfrac{du}{3}$

19. $\displaystyle\int \frac{e^{\frac{1}{x}}dx}{x^2} = -e^{\frac{1}{x}} + C$

let u = $\dfrac{1}{x}, du = -\dfrac{1}{x^2}$

20. $\displaystyle\int \left(1-2e^{\tan x}\right)\sec^2 x dx = \int \sec^2 x dx - 2\int e^{\tan x}\sec^2 x dx = \tan x - 2e^{\tan x} + C$

21. $\displaystyle\int 4e^{3\ln x}dx = 4\int e^{\ln x^3}dx = 4\int x^3 dx = 4\cdot\frac{x^4}{4} + C = x^4 + C$, note: $e^{\ln x} = x$

22. $\displaystyle\int \ln e^{2x}dx = \int 2x\ln e dx = \int 2x dx = 2\cdot\frac{x^2}{2} + C = x^2 + C$, $\ln e = 1$

23. $\displaystyle\int \sinh 3x dx = \frac{1}{3}\cosh 3x + C$

24. $\displaystyle\int x\coth x^2 dx = \frac{1}{2}\ln\sinh x^2 + C$

25. $\displaystyle\int e^{-x}\sinh x dx = \int e^{-x}\left(\frac{e^x - e^{-x}}{2}\right)dx = \frac{1}{2}\int\left(1-e^{-2x}\right)dx = \frac{x}{2} + \frac{e^{-2x}}{4} + C$

Logarithmic Integration

$$\int \frac{du}{u} = \ln u + C$$

Examples

1. $\displaystyle\int \frac{3dy}{4y-3} = \frac{3}{4}\int\frac{du}{u} = \frac{3}{4}\ln|4y-3| + C$

u = 4y – 3 , du = 4dy, $dy = \dfrac{du}{4}$

2. $\displaystyle\int \frac{x^2 dx}{x^3-4} = \frac{1}{3}\int\frac{du}{u} = \frac{1}{3}\ln\left|x^3-4\right| + C$

u = x^3 - 4, du = $3x^2$dx, $x^2dx = \frac{du}{3}$

3. $\int \frac{vdv}{4v^2-1} = \frac{1}{8}\int \frac{du}{u} = \frac{1}{8}\ln\left|4v^2-1\right| + C$

u = $4v^2$ – 1, du = 8vdv, $vdv = \frac{du}{8}$

4. $\int \frac{(2x-3)dx}{x^2-3x-8} = \int \frac{du}{u} = \ln\left|x^2-3x-8\right| + C$

u = x^2 – 3x – 8, du = (2x – 3)dx

5. $\int \frac{(y-4)dy}{y^2-8y+10} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln\left|y^2-8y+10\right| + C$

u = y^2 – 8y + 10, du = (2y – 8)dy= 2(y – 4) dy, $(y-4)dy = \frac{du}{2}$

6. $\int \frac{(1-3x)^2 dx}{x} = \int \frac{(1-6x+9x^2)dx}{x} = \int\left(\frac{1}{x} - 6 + 9x\right)dx = \ln x - 6x + \frac{9}{2}x^2 + C$

7. $\int \frac{(s^2+4)^2 ds}{s^3} = \int \frac{(s^4+8s^2+16)}{s^3}ds = \int\left(s + \frac{8}{s} + \frac{16}{s^3}\right)ds = \frac{1}{2}s^2 + 8\ln s - \frac{8}{s^2} + C$

8. $\int \frac{w+4}{w-1}dw = \int \frac{w-1+5}{w-1}dw = \int\left(1 + \frac{5}{w-1}\right)dw = \int dw + 5\int \frac{dw}{w-1} = w + 5\ln\left|w-1\right| + C$

9. $\int \tan x dx = \int \frac{\sin x}{\cos} dx = -\ln\left|\cos x\right| + C$

10. $\int \cot y dy = \int \frac{\cos y}{\sin y} dy = \ln\left|\sin y\right| + C$

11. $\int \frac{\cos z dz}{2+3\sin z} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln\left|2+\cos z\right| + C$

let u = 2 + 3 sin z, du = 3 cos z dz, $\cos z dz = \frac{du}{3}$

12. $\int \frac{\sin 2x dx}{4-3\cos 2x} = \frac{1}{6}\ln\left|4-3\cos 2x\right| + C$

let u= 4 – 3cos 2x, du = 6sin2xdx, $\sin 2x dx = \frac{du}{6}$

13. $\int \frac{\csc^2 w dw}{1+\cot w} = -\ln|1+\cot w| + C$

let u = 1 + cot w, du = -csc^2 wdw

14. $\int \frac{(x^2-2x)dx}{x^3-3x^2+4} = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|x^3-3x^2+4| + C$

let u = x^3 -3x^2 +4, du = (3x^2 – 6x)dx, $(x^2-2x)dx = \frac{du}{3}$

15. $\int \frac{x^3 dx}{xi1} = \int\left(x^2+x+1+\frac{1}{x-1}\right)dx = \frac{1}{3}x^3+\frac{1}{2}x^2+x+\ln|x-1|+C$

16. $\int \frac{\sec v \tan v dv}{4\sec v+3} = \frac{1}{4}\int \frac{du}{u} = \frac{1}{4}\ln|4\sec v+3| + C$

let u = 4sec v + 3, du = 4sec v tanv dv, $\sec v \tan v dv = \frac{du}{4}$

17. $\int \frac{e^x dx}{5e^x-4} = \frac{1}{5}\int \frac{du}{u} = \frac{1}{5}\ln|5e^x-4| + C$

let u = 5^{ex} – 4 , du = 5e^{x}dx, $e^x dx = \frac{du}{5}$

18. $\int \frac{e^{2t} dt}{3+e^{2t}} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|3+e^{2t}| + C$

let u = 3 + e^{2t}, du = 2^{e2t}dt, $e^{2t} dt = \frac{du}{2}$

19. $\int \frac{dx}{x\ln x} = \ln(\ln x) + C$

let u = ln x, $du = \frac{dx}{x}$

20. $\int \frac{\sin 2\theta d\theta}{1+\sin^2\theta} = \int \frac{2\sin\theta\cos\theta d\theta}{1+\sin^2\theta} = \ln(1+\sin^2\theta) + C$

let $u = 1+\sin^2\theta, du = 2\sin\theta\cos\theta d\theta$

21. $\int \frac{\sec^2 y \tan y dy}{4+\tan^2 y} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln(4+\tan^2 y) + C$

let $u = 4+\tan^2 y, du = 2\sec^2 y \tan y dy, \sec^2 y \tan y dy = \frac{du}{2}$

22. $\int \frac{(z+6)dz}{(z+2)^2} = \int \frac{(z+2+4)dz}{(z+2)^2} = \int \frac{(z+2)dz}{(z+2)^2} + 4\int \frac{dz}{(z+2)^2}$

$= \ln(z+2) - \frac{1}{z+2} + C$

23. $\int \frac{dx}{\sqrt{x}(1+\sqrt{x})} = 2\int \frac{du}{u} = 2\ln(1+\sqrt{x}) + C$

let $u = 1+\sqrt{x}, du = \frac{dx}{2\sqrt{x}} = 2du = \frac{dx}{\sqrt{x}}$

24. $\int \sec\psi d\psi = \int \sec\psi \cdot \left|\frac{\sec\psi + \tan\psi}{\sec\psi + \tan\psi}\right| d\psi = \ln|\sec\psi + \tan\psi| + C$

let $u = \sec\psi + \tan\psi, du = \sec\psi(\sec\psi + \tan\psi)d\psi$

25. $\int \csc\omega d\omega = \int \csc\omega \left|\frac{\csc\omega - \cot\omega}{\csc\omega - \cot\omega}\right| d\omega = \ln|\csc\omega - \cot\omega| + C$

let $u = \csc\omega - \cot\omega, du = (-\csc\omega\cot\omega + \csc^2\omega)d\omega = \csc\omega(\csc\omega - \cot\omega)d\omega$

26. $\int \frac{dx}{\sin 4x} = \int \csc 4x dx = \frac{1}{4}\ln|\csc 4x = \cot 4x| + C$

27. $\int \frac{dy}{\cos 5y} = \int \sec 5y dt = \frac{1}{5}\ln|\sec 5y + \tan 5y| + C$

28. $\int \frac{6+\sin 4z}{\sin 4z} dz = 6\int \frac{dz}{\sin 4z} + \int dz = 6\int \csc 4z dz + z + C = \frac{3}{2}\ln|\csc 4z - \cot 4z| + z + C$

29. $\int \frac{\cos x - \tan x}{\cos^2 x} dx = \int \sec x dx - \int \tan x \sec^2 x dx = \ln|\sec x + \tan x| - \frac{1}{2}\tan^2 x + C$

30. $\int \frac{dy}{\cos y \sin y} = 2\int \frac{dy}{\sin 2y} = 2\int \csc 2y dy = \ln|\csc 2y - \cot 2y| + C$

Integration of Trigonometric Functions

$$\int \cos ax\,dx = \frac{1}{a}\sin ax + C$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax + C$$

$$\int \csc^2 ax\,dx = -\frac{1}{a}\cot ax + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

$$\int \tan x\,dx = -\ln(\cos x) + C$$

$$\int \cot x\,dx = \ln(\sin x) + C$$

$$\int \csc x\,dx = \ln(\csc x - \cot x) + C$$

$$\int \sec x\,dx = \ln(\sec x + \tan x) + C$$

Examples

1. $\int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

2. $\int \cos(3x+4)\,dx = \int \cos u\,du = \frac{1}{3}\sin u + C = \frac{1}{3}\sin(3x+4) + C$

   let u = 3x +4, du = 3dx or $dx = \frac{du}{3}$

3. $\int \sin 4x\,dx = -\frac{1}{4}\cos 4x + C$

4. $\int x^2 \sin(x^3+4)\,dx = \int \sin(x^3+4)\,x^2\,dx = \frac{1}{3}\int \sin u\,du = -\frac{1}{3}\cos u + C = -\frac{1}{3}\cos(x^3+4) + C$

   let u = x^3 + 4, du = $3x^2$dx or $x^2 dx = \frac{du}{3}$

5. $\int e^{4x} \sin(e^{4x})\,dx = \int \sin(e^{4x})\,e^{4x}\,dx = \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(e^{4x}) + C$

   let u = e^{4x}, du =4e^{4x}dx or $e^{4x} dx = \frac{du}{4}$

6. $\int(\sin 3x-\cos 4x)dx=-\frac{1}{3}\cos 3x-\frac{1}{4}\sin 4x+C$

7. $\int(\sec^2 5x-\csc^2 6x)dx=\frac{1}{5}\tan 5x+\frac{1}{6}\cot 6x+C$

8. $\int\frac{\cos(\ln x)}{x}dx=\int\cos u du=\sin u+C=\sin(\ln x)+C$

let u = ln x, du = $\frac{dx}{x}$

9. $\int\frac{\cos^3 xdx}{1-\sin x}=\int\frac{\cos^2 x\cos xdx}{1-\sin x}=\int\frac{(1-\sin^2 x)\cos xdx}{1-\sin x}=\int\frac{(1-\sin x)(1+\sin x)\cos xdx}{1-\sin x}$

$=\int(1+\sin x)\cos xdx=\int\cos xdx+\int\sin x\cos xdx=\sin x+\frac{1}{2}\sin^2 x+C$

10. $\int\sin^3 y\cos ydy=\int u^3du=\frac{u^4}{4}+C=\frac{1}{4}\sin^4 y+C$

let u = sin y, du = cos y dy

11. $\int\cos^4 x\sin xdx=-\int u^4du=-\frac{1}{5}u^5+C=-\frac{1}{5}\cos^5 x+C$

let u = cos x, du = -sin x dx

12. $\int\frac{\cos xdx}{\sin^3 x}=\int u^{-3}du=-\frac{1}{2}u^{-2}+C=-\frac{1}{2\cos^2 x}+C$

let u = sin x, du = cos x dx

13. $\int\frac{\sin vdv}{\cos^2 v}=-\int\frac{du}{u^2}=\frac{1}{u}+C=\frac{1}{\cos v}+C=\sec v+C$

let u = cos x, du = - sin x dx

14. $\int\sin 3x\cos 3xdx=\frac{1}{3}\int udu=\frac{1}{3}\cdot\frac{u^2}{2}+C=\frac{1}{6}\sin^2 3x+C$

let u = sin 3x, du = 3cos 3x dx or $\frac{du}{3}=\cos 3xdx$

15. $\int\sin\frac{y-1}{3}\cos\frac{y-1}{3}dy=3\int udu=\frac{3}{2}u^2+C=\frac{3}{2}\sin\frac{y-1}{3}+C$

let u = $\sin\frac{y-1}{3}$, du = $\frac{1}{3}\cos\frac{y-1}{3}dy$ or $3du=\cos\frac{y-1}{3}$

16. $\int (8-3\sin 2\alpha)^4 \cos 2\alpha d\alpha = -\frac{1}{6}\int u^4 du = -\frac{1}{6}\cdot\frac{1}{5}u^5 + C = -\frac{1}{30}(8-3\sin 2\alpha)^5 + C$

let u = $8-3\sin 2\alpha$, du = $-6\cos 2\alpha d\alpha$ or $-\frac{du}{6} = \cos 2\alpha d\alpha$

17. $\int \frac{\cos 2t}{(1+\sin 2t)^4} dt = \frac{1}{2}\int u^{-4} du = \frac{1}{2}\cdot\frac{-1}{3}u^{-3} + C = -\frac{1}{6(1+\sin 2t)^3} + C$

let u = 1 + sin 2t, du = $2\cos 2t dt$ or $\cos 2t dt = \frac{du}{u}$

18. $\int \tan\theta \sec^2\theta d\theta = \int u du = \frac{u^2}{2} + C = \frac{1}{2}\tan^2\theta + C$

let $u = \tan\theta, du = \sec^2\theta d\theta$

or

$\int \tan\theta \sec^2\theta d\theta = \int \sec\theta(\sec\theta\tan\theta d\theta) = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}\sec^2\theta + C$

let $u = \sec\theta, du = \sec\theta\tan\theta d\theta$

19. $\int \sec^5 x \tan x dx = \int \sec^4 x(\sec x \tan x dx) = \int u^4 du = \frac{1}{5}u^5 + C = \frac{1}{5}\sec^5 x + C$

let $u = \sec x, du = \sec x \tan x dx$

20. $\int \tan^2(4x-1)\sec^2(4x-1)dx = \frac{1}{4}\int u^2 du = \frac{1}{4}\cdot\frac{1}{3}u^3 + C = \frac{1}{12}\tan(4x-1) + C$

let $u = \tan(4x-1), du = 4\sec^2(4x-1)dx, \sec^2(4x-1) = \frac{du}{4}$

21. $\int \sin^4(e^{2x})\cos(e^{2x})dx = \frac{1}{2}\int u^4 du = \frac{1}{2}\cdot\frac{u^5}{5} + C = \frac{1}{10}u^5 + C = \frac{1}{10}\sin^5(e^{2x}) + C$

let $u = \sin(e^{2x}), du = 2\cos(e^{2x}), \cos(e^{2x}) = \frac{du}{2}$

22.

$\int \frac{\sec^2 z}{\sqrt{1-4\tan z}} dz = -\frac{1}{4}\int \frac{du}{u^{1/2}} = -\frac{1}{4}\int u^{-1/2} du = -\frac{1}{4}\frac{u^{1/2}}{\frac{1}{2}} + C = -\frac{1}{2}u^{1/2} + C = -\frac{1}{2}(1-4\tan z) + C$

let $u = 1-4\tan z, du = -4\sec^2 z dz, \sec^2 z dz = -\frac{du}{4}$

23. $\int \frac{\sec y \tan y}{(3-8\sec y)^{1/2}} dy = -\frac{1}{8}\int u^{-1/2} du = -\frac{1}{8}\frac{u^{1/2}}{\frac{1}{2}} + C = -\frac{1}{4}(3-8\sec y)^{1/2} + C$

let $u = 3 - 8\sec y, du = -8\sec y \tan y dy, \sec y \tan y dy = \frac{du}{-8}$

24. $\int \ln \sec\alpha \tan\alpha d\alpha = \int u du = \frac{1}{2}u^2 + C = \frac{1}{2}(\ln \sec\alpha)^2 + C$

let $u = \ln \sec\alpha, du = \frac{\sec\alpha \tan\alpha}{\sec\alpha} d\alpha = \tan\alpha d\alpha$

25. $\int \frac{\sec^3 v \tan v}{(1+\sec^3 v)^4} dv = \frac{1}{3}\int u^{-4} du = \frac{1}{3}\cdot\frac{1}{-3}u^{-3} + C = -\frac{1}{9}(1-\sec^3 v)^{-3} + C$

let $u = 1 + \sec^3 v, du = 3\sec^2 v \sec v \tan v dv, \sec^3 v \tan v dv = \frac{du}{3}$

26. $\int \frac{dz}{e^z \sin^2(e^{-z})} = \int \csc^2(e^{-z}) e^{-z} dz = \cot(e^{-z}) + C$

let u = e^{-z}, du = $-e^{-z}$ dz

27. $\int \cot^2 4t dt = \int (\csc^2 4t - 1) dt = -\frac{1}{4}\cot 4t - t + C$

28. $\int \tan^2 6y dy = \int (\sec^2 6y - 1) dy = \frac{1}{6}\tan 6y - y + C$

29.

$\int \frac{\sin\theta d\theta}{\cos^4\theta} = \int \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos^3} d\theta = \int \tan\theta \sec^3\theta d\theta = \int \sec^2\theta(\sec\theta\tan\theta)d\theta = \frac{1}{3}\sec^3\theta + C$

30. $\int \frac{\cos^4 x dx}{\sin^6 x} = \int \frac{\cos^4 x}{\sin^4 x}\frac{dx}{\sin^2 x} = \int \cot^4 x \csc^2 x dx = -\frac{1}{5}\cot^5 x + C$

31. $\int (\tan\beta - 1)^2 d\beta = \int (\tan^2\beta - 2\tan\beta + 1) d\beta = \int ((\sec^2 - 1) - 2\tan\beta + 1) d\beta$

$= \int (\sec^2\beta - 2\tan\beta) d\beta = \tan\beta - 2\ln(\cos\beta) + C$

32.

$\int \frac{\cos^2\theta d\theta}{1-\sin\theta} = \int \frac{(1-\sin^2\theta)d\theta}{1-\sin\theta} = \int \frac{(1-\sin\theta)(1+\sin\theta)d\theta}{1-\sin\theta} = \int (1+\sin\theta)d\theta = \theta - \cos\theta + C$

33. $\int \frac{\sin^3 y dy}{1+\cos y} = \int \frac{\sin^2 y \sin y dy}{1+\cos y} \int \frac{(1-\cos^2 y)\sin y dy}{1+\cos y} = \int \frac{(1+\cos y)(1-\cos y)\sin y dy}{1+\cos y}$

$= \int (1-\cos y)\sin y dy = \int \sin y dy - \int \cos y \sin y dy = -\cos y + \frac{1}{2}\cos^2 y + C$

34. $\int \sin^2 w \csc^2 2w dw = \int \frac{\sin^2 w dw}{\sin^2 2w} = \int \frac{\sin^2 w dw}{4\sin^2 w \cos^2 w} = \frac{1}{4}\int \frac{dw}{\cos^2 w} = \frac{1}{4}\int \sec^2 w dw$
$= \frac{1}{4}\tan w + C$

35. $\int (\cos^4 x - \sin^4 x) dx = \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) dx = \int (\cos^2 x - \sin^2 x) dx$
$= \int \left[\frac{1}{2}(1+\cos 2x) - \frac{1}{2}(1-\cos 2x)\right] dx = \int \cos 2x dx = \frac{1}{2}\sin 2x + C$

36. $\int \sin 2x \sin 4x dx = \int \sin 2x (2\sin 2x \cos 2x) dx = 2\int \sin^2 2x \cos 2x dx = \frac{2}{3}\sin^3 2x + C$

37.
$\int \frac{(2\cos x - \sin x)^2 dx}{\sin x} = \int \frac{(4\cos^2 x - 4\cos x \sin x + \sin^2 x) dx}{\sin x} = \int \left[4\frac{(1-\sin^2 x)}{\sin x} - 4\cos x + \sin x\right] dx$
$= \int (4\csc x - 4\sin x - 4\cos x + \sin x) dx = \int (4\csc x - 3\sin x - 4\cos x) dx$
$= 4\ln(\csc x - \cot x) + 3\cos x - 4\sin x + C$

38. $\int \frac{1-\cos 2r}{1+\cos 2r} dr = \int \frac{\sin^2 r}{\cos^2 r} dr = \int \tan^2 r dr = \int (\sec^2 r - 1) dr = \tan r - r + C$

39. $\int \frac{\tan w dw}{1-\tan^2 w} = \int \frac{\frac{\sin w}{\cos w}}{1 - \frac{\sin^2 w}{\cos^2 w}} dw = \int \frac{\frac{\sin w}{\cos w}}{\frac{\cos^2 w - \sin^2 w}{\cos^2 w}} dw = \int \frac{\sin w \cos^2 w}{\cos w(\cos^2 w - \sin^2 w)} dw$
$= \frac{1}{2}\int \frac{\sin 2w}{\cos 2w} dw = \frac{1}{2}\int \tan 2w dw = \frac{1}{4}\ln\sec 2w + C$

40.
$\int \cos\alpha(1-\cos 2\alpha)^3 d\alpha = \int \cos\alpha[1-(\cos^2\alpha - \sin^2\alpha)]^3 d\alpha = \int \cos\alpha[1-\cos^2\alpha + (1-\cos^2\alpha)]^3 d\alpha$
$= \int \cos\alpha[2-2\cos^2\alpha]^3 d\alpha = \int \cos\alpha[2(1-\cos^2\alpha)]^3 d\alpha = \int \cos\alpha[8(1-\cos^2\alpha)^3] d\alpha$
$= 8\int \cos\alpha(1 - 3\cos^2\alpha + 3\cos^4\alpha - \cos^6\alpha) d\alpha$
$= 8\int \cos\alpha(1 - 3(1-\sin^2\alpha) + 3(1-\sin^2\alpha)^2 - (1-\sin^2\alpha)^3) d\alpha$

$= 8\int \cos\alpha(1 - 3 + 3\sin^2\alpha + 3 - 6\sin^2\alpha + 3\sin^4\alpha - 1 + 3\sin^2\alpha - 3\sin^4\alpha + \sin^6\alpha) d\alpha$
$= 8\int \sin^6\alpha \cos\alpha d\alpha = \frac{8}{7}\sin^7\alpha + C$

Transformation of Trigonometric Formulas

$$\frac{a}{b}\int \cos^n u \sin u du = \frac{-a}{b(n+1)du}\cos^{n+1} u + C$$
$$\frac{a}{b}\int \sin^n u \cos u du = \frac{a}{b(n+1)du}\sin^{n+1} u + C$$
$$\frac{a}{b}\int \tan^n u \sec^2 u du = \frac{a}{b(n+1)du}\tan^{n+1} u + C$$
$$\frac{a}{b}\int \cot^n u \csc^2 u du = \frac{-a}{b(n+1)du}\cot^{n+1} u + C$$
$$\frac{a}{b}\int \sec^n u \tan u du = \frac{a}{b(n)du}\sec^{n} u + C$$
$$\frac{a}{b}\int \csc^n u \cot u du = \frac{-a}{b(n)du}\csc^{n} u + C$$

If $\int \sin^m u \cos^n u du$ where ether m or n is a positive odd integer

If m is odd use $\sin^2 u = 1 - \cos^2 u$
If n is odd use $\cos^2 u = 1 - \sin^2 u$

If $\int \tan^m u \sec^n u du$ or $\int \cot^m u \csc^n u du$

m	n	Method	u	du
Even	Even	Transformation	tan/cot	$\sec^2/\csc^2$
Even	Odd	By parts		
Odd	Even	Transformation	Either	
Odd	Odd	Transformation	sec/csc	sectan/csccot

If $\int \sin^m u \cos u du$, when both m and n are positive even integers, use

$$\sin^2 u = \frac{1}{2}(1 - \cos 2u)$$
$$\cos^2 u = \frac{1}{2}(1 + \cos 2u)$$
$$\sin u \cos u = \frac{1}{2}\sin 2u$$

Examples

1. $\int \sin^3 v dv = \int \sin^2 v \sin v dv = \int (1 - \cos^2 v)\sin v dv = \int \sin v dv - \int \cos^2 v \sin v dv$

$$= -\cos v + \frac{1}{3}\cos^3 v + C$$

2. $\int \cos^3 y dy = \int \cos^2 y \cos y dy = \int (1-\sin^2 y)\cos y dy = \int \cos y dy - \int \sin^2 y \cos y dy$
$= \sin y - \frac{1}{3}\sin^3 y + C$

3.
$\int \cos^3 t \sin^3 t dt = \int \cos^3 t \sin^2 t \sin t dt = \int \cos^3 t(1-\cos^2 t)\sin t dt = \int (\cos^3 t - \cos^5 t)\sin t dt$
$= -\frac{1}{4}\cos^4 t + \frac{1}{6}\cos^6 t + C$

4.
$\int \cos^3 t \sin^3 t dt = \int \sin^3 t \cos^2 t \cos t dt = \int \sin^3 t(1-\sin^2 t)\cos t dt = \int (\sin^3 t - \sin^5 t)\cos t dt$

$= \frac{1}{4}\sin^4 t - \frac{1}{6}\sin^6 t + C$

5.
$\int \cos^3 t \sin^3 t dt = \int \left(\frac{1}{2}\sin 2t\right)^3 dt = \int \frac{1}{8}\sin^3 2t dt = \frac{1}{8}\int \sin^2 2t \sin 2t dt = \frac{1}{8}\int (1-\cos^2 2t)\sin 2t dt$
$= \frac{1}{8}\left[\int \sin 2t dt - \int \cos^2 2t \sin 2t dt\right] = \frac{1}{8}\left[-\frac{1}{2}\cos 2t + \frac{1}{6}\cos^3 2t\right] + C$

6. $\int \cos^2 2x \sin^3 2x dx = \int \cos^2 2x \sin^2 2x \sin 2x dx = \int \cos^2 2x(1-\cos^2 2x)\sin 2x dx$
$= \int \cos^2 2x \sin 2x dx - \int \cos^4 2x \sin 2x dx = -\frac{1}{6}\cos^3 2x + \frac{1}{10}\cos^5 2x + C$

7. $\int \sin^2 w \cos^5 w dw = \int \sin^2 w \cos^4 w \cos w dw = \int \sin^2 w(1-\sin^2 w)^2 \cos w dw$
$= \int \sin^2 w(1-2\sin^2 w + \sin^4 w)\cos w dw = \int (\sin^2 w - 2\sin^4 w + \sin^6 w)\cos w dw$
$= \int \sin^2 w \cos w dw - 2\int \sin^4 w \cos w dw + \int \sin^6 w \cos w dw$
$= \frac{1}{3}\sin^3 w - \frac{2}{5}\sin^5 w + \frac{1}{7}\sin^7 w + C$

8.
$\int \cos^5 3r dr = \int \cos^4 3r \cos 3r dr = \int (1-\sin^2 3r)^2 \cos 3r dr = \int (1-2\sin^2 3r + \sin^4 3r)\cos 3r dr$
$= \int \cos 3r dr - \int 2\sin^2 3r \cos 3r dr + \int \sin^4 3r \cos 3r dr = \frac{1}{3}\sin 3r - \frac{2}{9}\sin^3 3r + \frac{1}{15}\sin^5 3r + C$

9. $\int \frac{\sin^3 x dx}{\cos^6 x} = \int \frac{\sin^3 x}{\cos^3 x}\frac{1}{\cos^3 x} dx = \int \tan^3 x \sec^3 x dx = \int \sec^2 x \tan^2 x \sec x \tan x dx$

$$= \int \sec^2 (\sec^2 x - 1) \sec x \tan x dx = \int \sec^5 x \tan x dx - \int \sec^3 x \tan x dx$$

$$= \frac{1}{5} \sec^5 x - \frac{1}{3} \sec^3 x + C$$

10. $$\int \frac{\cos^3 x}{\sin^4 x} dx = \int \frac{\cos^3 x}{\sin^4 x} \frac{1}{\sin x} dx = \int \cot^3 x \csc x dx = \int \cot^2 x \cot x \csc x dx$$

$$= \int (\csc^2 x - 1) \cot x \csc x dx = \int \csc^3 x \cot x dx + \int \cot x \csc x dx$$

$$= -\frac{1}{3} \csc^3 x + \csc x + C$$

11 $$\int \sin^2 z \tan z dz = \int (1 - \cos^2 z) \frac{\sin z}{\cos z} dz = \int \frac{\sin z}{\cos z} dz - \int \frac{\cos^2 z \sin z}{\cos z} dz$$

$$= \int \tan z dz - \int \cos z \sin z dz = \ln|\sec z| - \frac{1}{2} \sin^2 z + C$$

12. $$\int \cos^2 v \cot v dv = \int \frac{(1 - \sin^2 v) \cos v}{\sin v} dv = \int \frac{\cos v}{\sin v} dv - \int \sin v \cos v dv$$

$$= \int \cot v dv - \frac{1}{2} \sin^2 v = \ln|\sin v| - \frac{1}{2} \sin^2 v + C$$

13. $$\int \sin^3 y (2 - 3\cos y)^2 dy = \int \sin^2 y (4 - 12\cos y + 9\cos^2 y) \sin y dy$$

$$= \int (1 - \cos^2 y)(4 - 12\cos y + 9\cos^2 y) \sin y dy$$

$$= \int (4 - 12\cos y + 9\cos^2 y - 4\cos^2 y + 12\cos^3 y - 9\cos^4 y) \sin y dy$$

$$= 4 \int \sin y dy - 12 \int \cos y \sin y dy + 5 \int \cos^2 y \sin y dy + 12 \int \cos^3 y \sin y dy$$

$$- 9 \int \cos^4 y \sin y dy$$

$$= -4\cos y + 6\cos^2 y - \frac{5}{3} \cos^3 y - 3\cos^4 y + \frac{9}{5} \cos^5 y + C$$

14. $$\int \frac{\sin^5 t dt}{\cos^2 t} = \int \frac{\sin^4 t \sin t dt}{\cos^2 t} = \int \frac{(1 - \cos^2)^2 \sin t dt}{\cos^2 t}$$

$$= \int \frac{(1 - 2\cos^2 t + \cos^4 t) \sin t dt}{\cos^2 t} = \int \frac{\sin t dt}{\cos^2 t} - 2 \int \sin t dt + \int \cos^2 t \cos t dt$$

$$= \int \tan t \sec t dt - 2 \int \sin t dt + \cos^2 t \sin t dt$$

$$= \sec t + 2\cos t - \frac{1}{3} \cos t + C$$

15. $$\int \frac{\cos^5 3x}{\sin^2 3x} dx = \int \frac{\cos^4 3x \cos 3x dx}{\sin^2 3x} = \int \frac{(1 - \sin^2 3x)^2 \cos 3x dx}{\sin^2 3x}$$

$$= \int \frac{\left(1-2\sin^2 3x+\sin^4 3x\right)\cos 3x dx}{\sin^2 3x} = \int \frac{\cos 3x dx}{\sin^2 3x} - 2\int \cos 3x dx + \int \sin^2 3x \cos 3x dx$$

$$= \int \cot 3x \csc 3x dx - 2\int \cos 3x dx + \int \sin^2 3x \cos 3x dx$$

$$= -\frac{1}{3}\csc 3x - \frac{2}{3}\sin 3x + \frac{1}{9}\sin^3 3x + C$$

16. $\int \cos^7 y dy = \int \cos^6 y \cos y dy = \int \left(1-\sin^2 y\right)^3 \cos y dy$

$$= \int \left(1-3\sin^2 y+3\sin^4 y+\sin^6 y\right)\cos y dy$$

$$= \int \cos y dy - 3\int \sin^2 y \cos y dy + 3\int \sin^4 y \cos y dy - \int \sin^6 y \cos y dy$$

$$= \sin y - \sin^3 y + \frac{3}{5}\sin^5 y - \frac{1}{7}\sin^7 y + C$$

17. $\int \sin^2 x dx = \frac{1}{2}\int (1-\cos 2x)dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + C$

18. $\int \cos^2 y dy = \frac{1}{2}\int (1+\cos 2x)dx = \frac{1}{x}\int dx + \frac{1}{2}\int \cos 2x dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C$

19. $\int \cos^2 z \sin^2 z dz = \int \left(\frac{1}{2}\sin 2z\right)^2 = \frac{1}{4}\int \frac{1}{2}(1-\cos 4z)dz = \frac{1}{8}\int dz - \frac{1}{8}\int \cos 4z dz$

$$= \frac{1}{8}z - \frac{1}{32}\sin 4z + C$$

20. $\int \sin^4 3u du = \int \left[\frac{1}{2}(1-\cos 6u)\right]^2 du = \frac{1}{4}\int \left(1-2\cos 6u+\cos^2 6u\right)du$

$$= \frac{1}{4}\int du - \frac{1}{2}\int \cos 6u du + \frac{1}{4}\cos^2 6u du = \frac{1}{4}u - \frac{1}{12}\sin 6u + \frac{1}{4}\int \frac{1}{2}(1+\cos 12u)du$$

$$= \frac{1}{4}u - \frac{1}{12}\sin 6u + \frac{1}{8}\int du + \frac{1}{8}\int \cos 12u du$$

$$= \frac{1}{4}u - \frac{1}{12}\sin 6u + \frac{1}{8}u + \frac{1}{96}\sin 12u + C$$

$$= \frac{3}{8}u - \frac{1}{12}\sin 6u + \frac{1}{96}\sin 12u + C$$

21. $\int \left(\cos^4 5v - \sin^4 5v\right)dv = \int \left(\cos^2 5v+\sin^2 5v\right)\left(\cos^2 5v-\sin^2 5v\right)dv$

$$= \int \cos 10v dv = \frac{1}{10}\sin 10v + C$$

22. $\int \sin^3 w \sin^3 2w dw = \int \sin^3 w \cdot 8\sin^3 w \cos^3 w dw = 8\int \sin^6 w \cos^2 w \cos w dw$

$$= 8\int \sin^6 \left(1-\sin^2 w\right)\cos w dw = 8\int \sin^6 w \cos w dw - 8\int \sin^8 w \cos w dw$$

$$= \frac{8}{7}\sin^7 w - \frac{8}{9}\sin^9 w + C$$

23. $\int \cos^2 \alpha \sin^3 2\alpha d\alpha = \int \cos^2 \alpha \cdot 8\sin^3 \alpha \cos^3 \alpha d\alpha = 8\int \cos^5 \alpha \sin^2 \alpha \sin \alpha d\alpha$

$$= 8\int \cos^5 \alpha\left(1-\cos^2 \alpha\right)\sin\alpha d\alpha = 8\int \cos^5 \alpha d\alpha - 8\int \cos^7 \alpha d\alpha$$

$$= -\frac{4}{3}\cos^6 \alpha + \cos^8 \alpha + C$$

24. $\int \sin^2 \beta \cos^4 \beta d\beta = \int \sin^2 \beta \cos^2 \beta \cos^2 \beta d\beta = \int \left(\frac{1}{2}\sin 2\beta\right)^2 \cdot \frac{1}{2}\left(1+\cos 2\beta\right)d\beta$

$$= \frac{1}{8}\int \sin^2 2\beta d\beta + \frac{1}{8}\int \sin^2 2\beta \cos 2\beta d\beta = \frac{1}{8}\int \frac{1}{2}\left(1-\cos 4\beta\right)d\beta + \frac{1}{48}\sin^3 2\beta + C$$

$$= \frac{1}{16}\int d\beta - \frac{1}{16}\int \cos 4\beta d\beta + \frac{1}{48}\sin^3 2\beta + C$$

$$= \frac{1}{16}\beta - \frac{1}{64}\sin 4\beta + \frac{1}{48}\sin^3 2\beta + C$$

25. $\int \cos^6 \frac{1}{2}x dx = \int \left[\frac{1}{2}\left(1+\cos x\right)\right]^3 dx = \frac{1}{8}\int (1+3\cos x + 3\cos^2 x + \cos^3 x)dx$

$$= \frac{1}{8}\int dx + \frac{3}{8}\int \cos x dx + \frac{3}{8}\int \frac{1}{2}\left(1+\cos 2x\right)dx + \frac{1}{8}\int \cos^2 x \cos x dx$$

$$= \frac{1}{8}x + \frac{3}{8}\sin x + \frac{3}{16}\int dx + \frac{3}{16}\int \cos 2x dx + \frac{1}{8}\int \left(1-\sin^2 x\right)\cos x dx$$

$$= \frac{1}{8}x + \frac{3}{8}\sin x + \frac{3}{16}x + \frac{3}{32}\sin 2x + \frac{1}{8}\sin x - \frac{1}{24}\sin^3 x + C$$

$$= \frac{5}{16}x + \frac{1}{2}\sin x + \frac{3}{32}\sin 2x - \frac{1}{24}\sin^3 x + C$$

26. $\int \cos^6 y \sin^6 y dy = \int \left[\frac{1}{2}\left(\sin 2y\right)\right]^6 dy = \frac{1}{64}\int \left[\frac{1}{2}\left(1-\cos 4y\right)\right]^3 dy$

$$= \frac{1}{512}\int \left(1-3\cos 4y + 3\cos^2 4y + \cos^3 4y\right)dy$$

$$= \frac{1}{512}\int dy - \frac{3}{512}\int \cos 4y dy + \frac{3}{512}\int \frac{1}{2}\left(1+\cos 8y\right)dy - \frac{1}{512}\int \cos^2 4y \cos 4y dy$$

$$= \frac{1}{512}y - \frac{3}{2048}\sin 4y + \frac{3}{1024}\int dy + \frac{3}{1024}\int \cos 8y dy - \int \frac{1}{512}\int \left(1-\sin^2 4y\right)\cos 4y dy$$

$$= \frac{1}{512}y - \frac{3}{2048}\sin 4y + \frac{3}{1024}y + \frac{3}{8192}\sin 8y - \frac{1}{2048}\sin 4y + \frac{1}{6144}\sin^3 4y + C$$
$$= \frac{5}{1024}y - \frac{1}{512}\sin 4y + \frac{3}{8192}\sin 8y + \frac{1}{6144}\sin^3 4y + C$$

27. $\int \tan^3 \phi d\phi = \int \tan^2 \phi \tan \phi d\phi = \int (\sec^2 \phi - 1)\tan \phi d\phi$
$$= \int \sec^2 \phi \tan \phi d\phi - \int \tan \phi d\phi = \int \sec \phi \sec \phi \tan \phi d\phi - \ln|\sec \phi| + C$$
$$= \frac{1}{2}\sec^2 \phi - \ln|\sec \phi + C|$$

28. $\int \cot^4 \varphi d\varphi = \int (\csc^2 \varphi - 1)^2 d\varphi = \int (\csc^4 \varphi - 2\csc^2 \varphi + 1)d\varphi$
$$= \int \csc^2 \varphi \csc^2 \varphi d\varphi - 2\int \csc^2 \varphi d\varphi + \int d\varphi$$
$$= \int (\cot^2 \varphi + 1)\csc^2 \varphi d\varphi + 2\cot \varphi + \varphi + C$$
$$= \int \cot^2 \varphi \csc^2 \varphi d\varphi + \int \csc^2 \varphi d\varphi + 2\cot \varphi + \varphi + C$$
$$= -\frac{1}{3}\cot^3 \varphi - \cot \varphi + 2\cot \varphi + \varphi + C$$
$$= -\frac{1}{3}\cot^3 \varphi + \cot \varphi + \varphi + C$$

29. $\int \cot^5 2\gamma d\gamma = \int \cot^4 2\gamma \cot 2\gamma d\gamma = \int (\csc^2 2\gamma - 1)^2 \cot 2\gamma d\gamma$
$$= \int (\csc^4 2\gamma - 2\csc^2 2\gamma + 1)\cot 2\gamma d\gamma$$
$$= \int \csc^4 2\gamma \cot 2\gamma d\gamma - 2\int \csc^2 2\gamma \cot 2\gamma d\gamma + \int \cot 2\gamma d\gamma$$
$$= -\frac{1}{8}\csc^4 2\gamma + \frac{1}{2}\csc^2 2\gamma + \frac{1}{2}\ln|\sin 2\gamma| + C$$

30. $\int \sec^4 \lambda d\lambda = \int \sec^2 \lambda \sec^2 \lambda d\lambda = \int (\tan^2 \lambda + 1)\sec^2 \lambda d\lambda$
$$= \int \tan^2 \lambda \sec^2 \lambda d\lambda + \int \sec^2 \lambda d\lambda$$
$$= \frac{1}{3}\tan^3 \lambda + \tan \lambda + C$$

31. $\int \csc^4 3\varpi d\varpi = \int \csc^2 3\varpi \csc^2 3\varpi d\varpi = \int (\cot^2 3\varpi + 1)\csc^2 3\varpi d\varpi$
$$= \int \cot^2 3\varpi \csc^2 3\varpi d\varpi + \int \csc^2 3\varpi d\varpi$$
$$= -\frac{1}{9}\cot^3 3\varpi - \frac{1}{3}\cot 3\varpi + C$$

32. $\int \sec^2 4\theta \tan 4\theta d\theta = \frac{1}{8}\tan^2 4\theta + C$

33. $\int \sec^4(3x+1)\tan^3(3x+1)dx = \int \sec^4(3x+1)\tan^2(3x+1)\tan(3x+1)dx$

$$= \int \sec^4(3x+1)\left[\sec^2(3x+1)-1\right]\tan(3x+1)dx$$

$$= \int \sec^6(3x+1)\tan(3x+1)dx - \int \sec^4(3x+1)\tan(3x+1)dx$$

$$= \frac{1}{18}\sec^6(3x+1) - \frac{1}{12}\sec^4(3x+1) + C$$

34. $\int \sec^4 5y \tan^4 5y dy = \int \sec^2 5y \sec^2 5y \tan^4 5y dy$

$$= \int \sec^2 5y\left(\tan^2 5y + 1\right)\tan^4 5y dy = \int \left(\tan^6 5y + \tan^4 5y\right)\sec^2 5y dy$$

$$= \int \tan^6 5y \sec^2 5y dy + \int \tan^4 5y \sec^2 5y dy$$

$$= \frac{1}{35}\tan^7 5y + \frac{1}{25}\tan^5 5y + C$$

35. $\int \csc^6 4z dz = \int \csc^4 4z \csc^2 4z dz = \int \left(\cot^2 4z + 1\right)^2 \csc^2 4z dz$

$$= \int \left(\cot^4 4z + 2\cot^2 4z + 1\right)\csc^2 4z dz$$

$$= \int \cot^4 4z \csc^2 4z dz + 2\int \cot^2 4z \csc^2 4z dz + \int \csc^2 4z dz$$

$$= -\frac{1}{20}\cot^5 4z - \frac{1}{6}\cot^3 4z - \frac{1}{4}\cot + C$$

Integration Leading to Inverse Trigonometric Functions

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin\frac{u}{a} + C$$

Examples

1. $\int \frac{dx}{9 + x^2} = \frac{1}{3}\arctan\frac{x}{3} + C$

   let u = x, du = dx, a = 3

2. $\int \frac{dx}{3 + x^2} = \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C$

   let u = x, du = dx, $a = \sqrt{3}$

3. $\int \frac{dx}{9+4x^2} = \frac{1}{2}\cdot\frac{1}{3}\arctan\frac{2x}{3} + C = \frac{1}{6}\arctan\frac{2x}{3} + C$

let u = 2x, du = 2dx, $dx = \frac{du}{2}, a = 3$

4. $\int \frac{dx}{25+(3x-1)^2} = \frac{1}{3}\cdot\frac{1}{5}\arctan\frac{3x-1}{5} + C = \frac{1}{15}\arctan\frac{3x-1}{5} + C$

let u = 3x – 1 , du = 3dx, $dx = \frac{du}{3}, a = 5$

5. $\int \frac{xdx}{2+(3x^2+4)^2} = \frac{1}{6}\cdot\frac{1}{\sqrt{2}}\arctan\frac{3x^2+4}{\sqrt{2}} + C$

let u = $3x^2$ +4, du = 6xdx, $xdx = \frac{du}{6}, a = \sqrt{2}$

6. $\int \frac{dy}{\sqrt{4-y^2}} = \arcsin\frac{y}{2} + C$

let u = y, du =dy, a = 2

7. $\int \frac{ydy}{\sqrt{4-y^2}} = -\frac{1}{2}\int (4-y^2)^{-1/2}\, ydy = -(4-y^2)^{1/2} + C$

8. $\int \frac{ydy}{\sqrt{4-y^4}} = \frac{1}{2}\arcsin\frac{y^2}{2} + C$

let u = y^2, du = 2ydy, $ydy = \frac{du}{2}$, a = 2

9. $\int \frac{dy}{\sqrt{25-4y^2}} = \frac{1}{2}\arcsin\frac{2y}{5} + C$

let u = 2y, du = 2dy, $dy = \frac{du}{2}$, a= 5

10. $\int \frac{dy}{\sqrt{5-(3y+2)^2}} = \frac{1}{3}\arcsin\frac{3y+2}{\sqrt{5}} + C$

let u = 3y +2, du = 3dy, $dy = \frac{du}{3}, a = \sqrt{5}$

11. $\int \frac{ydy}{\sqrt{4-\left(3y^2-1\right)^2}} = \frac{1}{6}\arcsin\frac{3y^2-1}{2} + C$

let u = 3y^2 – 1, du = 6ydy, $ydy = \frac{du}{6}, a = 2$

12. $\int \frac{dz}{z^2+6z+11} = \int \frac{dz}{\left(z^2+6z+9\right)+2} = \int \frac{dz}{(z+3)^2+2} = \frac{1}{\sqrt{2}}\arctan\frac{z+3}{\sqrt{2}} + C$

let u = z + 3, du = dz, $a = \sqrt{2}$

13. $\int \frac{dp}{4p^2-4p+5} = \int \frac{dp}{4p^2-4p+1+4} = \int \frac{dp}{(2p-1)^2+4} = \frac{1}{4}\arctan\frac{2p-1}{2} + C$

let u = 2p – 1, du = 2dp, $dp = \frac{du}{2}, a = 2$

14. $\int \frac{\sin q}{3+\cos^2 q}dq = -\frac{1}{\sqrt{3}}\arctan\frac{\cos q}{\sqrt{3}} + C$

let u = cos q, du = –sin q dq

15. $\int \frac{dr}{6r-4-9r^2} = -\int \frac{dr}{9r^2-6r+1+3} = -\int \frac{dr}{(3r-1)^2+3} = -\frac{1}{3\sqrt{3}}\arctan\frac{3r-1}{\sqrt{3}} + C$

let u = 3r – 1, du = 3dr, $dr = \frac{du}{3}, a = \sqrt{3}$

16. $\int \frac{(y+3)dy}{y^2+16} = \int \frac{ydy}{y^2+16} + 3\int \frac{dy}{y^2+16} = \frac{1}{2}\ln\left(y^2+16\right) + \frac{3}{4}\arctan\frac{y}{4} + C$

17. $\int \frac{\left(y^2+1\right)dy}{y^2+4} = \int \left(1-\frac{3}{y^2+4}\right)dy = y - \frac{3}{2}\arctan\frac{y}{2} + C$

18. $\int \frac{e^{2w}dw}{\sqrt{9-4e^{4w}}} = \frac{1}{4}\arcsin\frac{2e^{2w}}{3} + C$

let u =2e ^{2w}, $du = 4e^{2w}dw, e^{2w}dw = \frac{du}{4}, a = 3$

19. $\int \frac{dt}{\sqrt{15+4t-4t^2}} = \int \frac{dt}{\sqrt{15+1-(1-4t+4t^2)}} = \int \frac{dt}{\sqrt{16-(1-2t)^2}} = -\frac{1}{2}\arcsin\frac{1-2t}{4} + C$

let u = 1 – 2t, $du = -2dt, dt = -\frac{du}{2}, a = 4$

20. $\int \frac{\sec^2 w dw}{\sqrt{5-\sec^2 w}} = \int \frac{\sec^2 w dw}{\sqrt{5-(\tan^2 w+1)}} = \int \frac{\sec^2 w dw}{\sqrt{4-\tan^2 w}} = \arcsin \frac{\tan w}{2} + C$

let u = tan w, du = $\sec^2 w dw$, a = 2

21. $\int \frac{du}{\sqrt{9e^{-2u}-1}} = \int \frac{du}{\sqrt{\frac{9}{e^{2u}}-1}} = \int \frac{du}{\sqrt{\frac{9-e^{2u}}{e^{2u}}}} \int \frac{du}{\frac{\sqrt{9-e^{2u}}}{e^u}} = \int \frac{e^u du}{\sqrt{9-e^{2u}}} = \arcsin \frac{e^u}{3} + C$

22. $\int \frac{xdx}{\sqrt{-7-8x-x^2}} = \int \frac{(x+4-4)dx}{\sqrt{-7-8x-x^2}} = \int \frac{(x+4)dx}{\sqrt{-7-8x-x^2}} - 4\int \frac{dx}{\sqrt{9-(x^2+8x+7+9)}}$

$= -\frac{1}{2}\int \frac{(x+4)dx}{\sqrt{-7-8x-x^2}} - 4\int \frac{dx}{\sqrt{9-(x+4)^2}} = -\sqrt{-7-8x-x^2} - 4\arcsin \frac{x+4}{3} + C$

23. $\int \frac{dx}{\sqrt{2ax-x^2}} = \int \frac{dx}{\sqrt{a^2-(x^2-2ax+a^2)}} = \int \frac{dx}{\sqrt{a^2-(x-a)^2}} = \arcsin \frac{x-a}{a} + C$

24. $\int \frac{(1+\tan v)dv}{\cos^2 v\sqrt{5-3\tan^2 v}} = \int \frac{\sec^2 v dv}{\sqrt{5-3\tan^2 v}} + \int \frac{\sec^2 v \tan v dv}{\sqrt{5-3\tan^2 v}}$

$= \frac{1}{\sqrt{3}} \arcsin \frac{\sqrt{3}\tan v}{\sqrt{5}} - \frac{1}{3}\sqrt{5-3\tan^2 v} + C$

25. $\int \frac{ds}{s\sqrt{s^2-a^2}} = \int \frac{ds}{s^2\sqrt{1-\frac{a^2}{s^2}}} = -\frac{1}{a}\int \frac{du}{\sqrt{a^2-u^2}} = -\frac{1}{a}\arcsin \frac{a}{s} + C$

let u = $\frac{a}{s}, du = -\frac{a}{s^2}ds, \frac{ds}{s^2} = -\frac{du}{a}$

Integration by Trigonometric Substitution

1. $\int \frac{x^2 dx}{\sqrt{25-x^2}}$

Solution.

Substitute

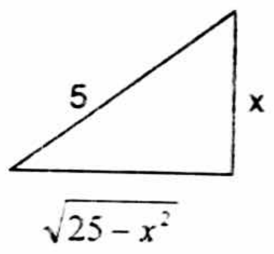
x = 5 sin θ or $\sin\theta = \frac{x}{5}$, $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$, θ = $\sin^{-1} \frac{x}{5}$

$dx = 5 \cos \theta \, d\theta$,

$25 - x^2 = 25(1 - \sin^2 \theta) = 25 \cos^2 \theta$,

or $\cos\theta = \dfrac{\sqrt{25 - x^2}}{5}$

As shown from the figure,



then by substitution,

$$\int \frac{x^2 dx}{\sqrt{25 - x^2}} = \int \frac{25\sin^2\theta \cdot 5\cos\theta d\theta}{5\cos\theta}$$
$$= 25 \int \sin^2 \theta d\theta$$

Using the identity ,

$$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\int \sin^2 \theta d\theta = \frac{1}{2}\int (1 - \cos 2\theta) d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$
$$= \frac{1}{2}(\theta - \sin\theta\cos\theta) + C$$

Substituting this from the above,

$$\int \frac{x^2 dx}{\sqrt{25 - x^2}} = \frac{25}{2}(\theta - \sin\theta\cos\theta) + C$$
$$= \frac{25}{2}\left[\sin^{-1}\frac{x}{5} - \frac{x\sqrt{25 - x^2}}{25}\right] + C$$

2. $\displaystyle\int \frac{dx}{\sqrt{9 + x^2}}$

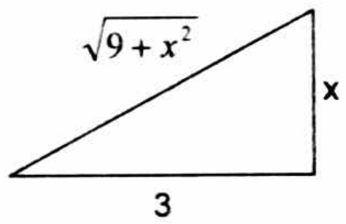
Solution. Let x = 3 tan θ, dx = 3sec² θdθ

$$\sqrt{9+x^2} = \sqrt{9+9\tan^2\theta}$$
$$= 3\sec\theta$$

$$\int\frac{dx}{\sqrt{9+x^2}} = \int\frac{3\sec^2\theta d\theta d}{3\sec\theta}$$
$$= \int \sec\theta d\theta$$
$$= \ln\left|\sec\theta + \tan\theta\right| + C$$

Since tan θ = $\frac{x}{3}$, then sec θ = $\frac{\sqrt{9+x^2}}{3}$

As shown in the figure,



Thus,

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

Integration of Rational Functions of Sin x and Cosine x and Other Trigonometric Integrals

1. $\int\frac{dx}{1+\cos x}$

Solution.

$$\int \frac{dx}{1+\cos x} = \int \frac{\frac{2dz}{1+z^2}}{1+\frac{1-z^2}{1+z^2}}$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{1+z^2+1-z^2}{1+z^2}} = \int \frac{2dz}{2} = \int dz$$

$$= z + C$$

$$= \tan\frac{x}{2} + C$$

2. $\int \frac{dx}{2+\sin x}$

Solution.

$$\int \frac{dx}{2+\sin x} = \int \frac{\frac{2dz}{1+z^2}}{2+\frac{2z}{1+z^2}}$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{2(1+z^2)+2z}{1+z^2}}$$

$$= \int \frac{2dz}{2\left(z^2+z+1\right)}$$

$$= \int \frac{dz}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$= \int \frac{du}{u^2+a^2}; \left[u = z+\frac{1}{2}, a = \frac{\sqrt{3}}{2}\right]$$

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + C$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1+2\tan\frac{x}{2}}{\sqrt{3}} + C$$

3. $\int \sec x dx$

Solution

$$\int \sec x dx = \int \frac{dx}{\cos x}$$

$$= \int \frac{2dz}{1+z^2} \cdot \frac{1+z^2}{1-z^2} = \int \frac{2dz}{1-z^2}$$

By method of partial fractions :

$$\frac{2}{1-z^2} = \frac{A}{1-z} + \frac{B}{1+z}$$

$$2 = A(1+z) + B(1-z)$$

$$= (A+B) + (A-B)z$$

$$A+B = 2, A-B = 0$$

A =B = 1

$$\int \frac{2dz}{1-z^2} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z} = \ln|1+z| - \ln|1-z| + C$$

$$= \ln\left|\frac{1+z}{1-z}\right| + C$$

$$= \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C$$

$$= \ln\left|\frac{\tan\frac{\pi}{4} + \tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right| + C$$

$$= \ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$$

4. $\int \frac{dx}{1-\sin x+\cos x}$

Solution

$$\int \frac{dx}{1-\sin x+\cos x} = \int \frac{\frac{2dz}{1+z^2}}{1-\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}} = 2\int \frac{dz}{(1+z^2)-2z+(1-z^2)}$$

$$= 2\int \frac{dz}{2-2z} = \int \frac{dz}{1-z}$$

$$= -\ln|1-z| + C$$

$$= -\ln\left|1-\tan\frac{x}{2}\right| + C$$

Integration by Parts

$$\int u dv = uv - \int v du$$

Examples

1. $\int xe^x dx$

let u = x, du = dx, dv = e^xdx, v = e^x

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + C$$

2. $\int ye^{-3y} dy$

let u = y, du = dy, dv = e^{-3y}dy, $v = -\frac{1}{3}e^{-3y}$

$$\int ye^{-3y} dy = -\frac{1}{3}ye^{-3y} + \frac{1}{3}\int e^{-3y} dy = -\frac{1}{3}ye^{-3y} - \frac{1}{9}e^{-3y} + C$$

3. $\int x\cos 2x dx$

let u = x, du = dx, dv = cos2sdx, $v = \frac{1}{2}\sin 2x$

$$\int x\cos 2x dx = \frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x dx = \frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x + C$$

4. $\int y^2 \sin 4y dy$

let u = y^2, du = 2ydy, dv = sin4ydy, $v = -\frac{1}{4}\cos 4y$

$$\int y^2 \sin 4y dy = -\frac{1}{4}y^2 \cos 4y + \frac{1}{2}\int y\cos 4y dy$$

let u' = y, du' = dy, dv' = cos4ydy, $v' = \frac{1}{4}\sin 4y$

$$= -\frac{1}{4}y^2 \cos 4y + \frac{1}{2}\left(\frac{1}{4}y\sin 4y - \frac{1}{4}\int \sin 4y dy\right)$$

$$= -\frac{1}{4}y^2 \cos 4y + \frac{1}{8}y\sin 4y + \frac{1}{32}\cos 4y + C$$

5. $\int z^2 \cos z dz$

let u = z^2, du = 2zdz, dv = cos zdz, v = sin z

$$\int z^2 \cos z dz = z^2 \sin z - 2\int z\sin z dz$$

let u' = z, du' = dz, dv' = sinzdz, v' = -cos z

$$= z^2 \sin z - 2\left(- z\cos z + \int \cos z dz\right)$$
$$= z^2 \sin z + 2z\cos z - 2\sin z + C$$

6. $\int \ln x dx$

let u = ln x, $du = \frac{dx}{x}, dv = dx, v = x$

$$\int \ln x dx = x\ln x - \int x dx = x\ln x - \frac{1}{2}x^2 + C$$

7. $\int w\ln w dw = \frac{1}{2}w^2 \ln w - \frac{1}{2}\int w dw = \frac{1}{2}w^2 \ln w - \frac{1}{4}w^2 + C$

let u = ln w, $du = \frac{dw}{w}, dv = w, v = \frac{1}{2}w^2$

$$\int w\ln w dw = \frac{1}{2}w^2 \ln w - \frac{1}{2}\int w dw = \frac{1}{2}w^2 \ln w - \frac{1}{4}w^2 + C$$

8. $\int r^2 e^{-r} dr$

let u = r^2, du = 2rdr, dv = e^{-r}dr, v = $-e^{-r}$

$$\int r^2 e^{-r} dr = -r^2 e^{-r} + 2\int re^{-r} dr$$
$$= -r^2 e^{-r} - 2re^{-r} + 2\int e^{-rdr}$$
$$= -r^2 e^{-r} - 2re^{-r} - 2e^{-r} + C$$

9. $\int s^2 e^{2s} ds$

let u = s^2, du = 2sds, dv = e^{2s}ds, $v = \frac{1}{2}e^{2s}$

$$\int s^2 e^{2s} ds = \frac{1}{2}s^2 e^{2s} - \int se^{2s} ds$$

let u' = s, du' = ds, dv' = e^{2s}ds, $v' = \frac{1}{2}e^{2s}$

$$= \frac{1}{2}s^2 e^{2s} - \left(\frac{1}{2}se^{2s} - \int e^{2s} ds\right)$$
$$= \frac{1}{2}s^2 e^{2s} - \frac{1}{2}se^{2s} + \frac{1}{4}e^{2s} + C$$

10. $\int \tan^{-1} x dx$

let u = $\tan^{-1}$x, $du = \frac{dx}{1+x^2}$, dv = dx, v = x

$$\int \tan^{-1} x dx = x\tan^{-1} x - \int \frac{x dx}{1+x^2} = x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

11. $\int y \tan^{-1} y dy$

let u = $\tan^{-1}$ y, $du = \dfrac{dy}{1+y^2}, dv = dy, v = \dfrac{1}{2}y^2$

$$\int y \tan^{-1} y dy = \frac{1}{2}y^2 \tan^{-1} y - \frac{1}{2}\int \frac{y^2 dy}{1+y^2} = \frac{1}{2}y^2 \tan^{-1} y - \frac{1}{2}\int\left(1 - \frac{1}{1+y^2}\right)dy$$

$$= \frac{1}{2}y^2 \tan^{-1} y - \frac{1}{2}\int dy + \int \frac{dy}{1+y^2} = \frac{1}{2}y^2 \tan^{-1} y - \frac{1}{2}y + \tan^{-1} y + C$$

12. $\int x^3 \sin(x^2)dx =$

let u = x^2, du = 2xdx, dv = $x\sin x^2$dx, $v = -\dfrac{1}{2}\cos x^2$

$$\int x^3 \sin(x^2)dx = -\frac{1}{2}x^2 \cos x^2 + \int x \cos x^2 dx$$

$$= -\frac{1}{2}x^2 \cos x^2 \frac{1}{2}\sin x^2 + C$$

13. $\int x(2x-1)dx$

let u = x, du = dx, dv = $(2x - 1)^7$dx, $v = \dfrac{1}{8}(2x-1)$

$$\int x(2x-1)dx = \frac{1}{16}x(2x-1)^8 - \frac{1}{16}\int (2x-1)^8 dx$$

$$= \frac{1}{16}x(2x-1)^8 - \frac{1}{288}(2x-1)^9 + C$$

14. $\int \dfrac{ydy}{(y-1)^5}$

let u = y, du = dy, dv = $(y-1)^{-5} dy$, $v = -\dfrac{1}{4}(y-1)^{-4}$

$$\int \frac{ydy}{(y-1)^5} = -\frac{1}{4}y(y-1)^{-4} + \frac{1}{4}\int (y-1)^{-4} dy$$

$$= -\frac{y}{4(y-1)^4} - \frac{1}{12}(y-1)^{-3} + C$$

$$= -\frac{y}{4(y-1)^4} - \frac{1}{12(y-1)^3} + C$$

15. $\int x\sqrt{x+4}dx$

let u = x, du = dx, dv = $(x + 4)^{1/2}$dx, $v = \dfrac{2}{3}(x+4)^{3/2}$

$$\int x\sqrt{x+4}dx = \frac{2}{3}x(x+4)^{3/2} - \frac{2}{3}\int(x+4)dx$$
$$= \frac{2}{3}x(x+4)^{3/2} - \frac{4}{15}(x+4)^{5/2} + C$$

16. $\int x^3(a^2+x^2)^{1/2}dx$

let u = x^2, du 2xdx, dv = x (a^2 + x^2)$^{1/2}$dx, $v = \frac{1}{3}(a^2+x^2)^{3/2}$

$$\int x^3(a^2+x^2)^{1/2}dx = \frac{1}{3}x^2(a^2+x^2)^{3/2} - \frac{1}{3}\int x(a^2+x^2)^{1/2}dx$$
$$= \frac{1}{3}x^2(a^2+x^2)^{3/2} - \frac{2}{15}(a^2+x^2)^{5/2} + C$$
$$= \frac{1}{15}(a^2+x^2)^{3/2}(3x^2-2a^2) + C$$

17. $\int y^3(a^2-y^2)dy$

let u = y^2, du = 2ydy, dv = y(a^2 – y^2)$^{1/2}$dy, $v = -\frac{1}{3}(a^2-y^2)^{3/2}$

$$\int y^3(a^2-y^2)^{1/2}dy = -\frac{1}{3}y^2(a^2-y^2)^{3/2} + \frac{2}{3}\int y(a^2-y^2)^{1/2}dy$$
$$= -\frac{1}{3}y^2(a^2-y^2)^{3/2} - \frac{2}{15}(a^2-y^2)^{5/2} + C$$
$$= -\frac{1}{3}y^2(a^2-y^2)^{3/2}(3y^2+2a^2) + C$$

18. $\int x^3e^{-x^2}dx$

let u = x^2, du = 2xdx, $dv = xe^{-x^2}, v = -\frac{1}{2}e^{-x^2}$

$$\int x^3e^{-x^2}dx = -\frac{1}{2}x^2e^{-x^2} + \int xe^{-x^2}dx$$
$$= -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2} + C$$
$$= -\frac{1}{2}x^2e^{-x^2}(x^2+1) + C$$

19. $\int \alpha\sin^2\alpha d\alpha$

$$\int \alpha\sin^2\alpha d\alpha = \int \alpha\left[\frac{1}{2}(1-\cos 2\alpha)\right]d\alpha = \frac{1}{2}\int \alpha d\alpha - \frac{1}{2}\int \alpha\cos 2\alpha d\alpha$$

let $u = \alpha, du = d\alpha, dv = \cos 2\alpha d\alpha, v = \frac{1}{2}\sin 2\alpha$

$$= \frac{1}{4}\alpha^2 - \frac{1}{2}\left(\frac{1}{2}\alpha \sin 2\alpha - \frac{1}{2}\int \sin 2\alpha d\alpha\right)$$
$$= \frac{1}{4}\alpha^2 - \frac{1}{4}\alpha \sin 2\alpha + \frac{1}{8}\cos 2\alpha + C$$

20. $\int x \sin 4x \cos 4x dx$

let u = x, du = dx, dv = sin 4xcos 4xdx, $v = \frac{1}{8}\sin^2 4x$

$$\int x \sin 4x \cos 4x dx = \frac{1}{8} x \sin^2 4x - \frac{1}{8}\int \sin^2 4x dx$$
$$= \frac{1}{8} x \sin^2 4x - \frac{1}{8}\int \left[\frac{1}{2}(1 - \cos 8x)\right] dx$$
$$= \frac{1}{8} x \sin^2 4x - \frac{1}{16}\int dx - \frac{1}{16}\int \cos 8x dx$$
$$= \frac{1}{8} x \sin^2 4x - \frac{x}{16} + \frac{1}{28}\sin 8x + C$$

21. $\int y \sec^2 dy$

let u = y, du = dy, dv = sec^{2}ydy, v = tan y

$$\int y \sec^2 dy = y \tan y - \int \tan y dy = y \tan y - \ln \cos y + C$$

22. $\int z \csc^2 z \cot z dz$

let u = z, du = dz, dv = csc z cot z dz, $v = -\frac{1}{2}\csc^2 z$

$$\int z \csc^2 z \cot z dz = -\frac{1}{2} z \csc^2 z + \frac{1}{2}\int \csc^2 z dz$$
$$= -\frac{1}{2} z \csc^2 z - \frac{1}{2}\cot z + C$$

23. $\int w \cosh \frac{w}{a} dw$

let u = w, du = dw, $dv = \cosh \frac{w}{a} dw, v = a \sinh \frac{w}{a}$

$$\int w \cosh \frac{w}{a} dw = aw \sinh \frac{w}{a} - a \int \sinh \frac{w}{a} dw$$
$$= aw \sinh \frac{w}{a} - a^2 \cosh \frac{w}{a} + C$$

24. $\int t \sinh \frac{t}{a} dt$

let u = t, du = dt, $dv = \sinh \frac{t}{a} dt, v = a\cosh \frac{t}{a}$

$$\int t \sinh \frac{t}{a} dt = at\cosh \frac{t}{a} - a\int \cosh \frac{t}{a} dt$$

$$at\cosh \frac{t}{a} - a^2 \sinh \frac{t}{a} + C$$

25. $\int \csc^3 x dx$

let $u = \csc x, du = -\csc x \cot x dx, dv = \csc^2 x dx, v = -\cot x$

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc x \cot^2 x dx$$

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc x(\csc^2 x - 1)dx$$

$$\int \csc^3 x dx = -\csc x \cot x - \int \csc^3 x dx + \int \csc x dx$$

$$2\int \csc^3 x dx = -\csc x \cot x + \int \csc x dx$$

$$2\int \csc^3 x dx = -\csc x \cot x + \ln|\csc x - \cot x| + C$$

$$\int \csc^3 x dx = \frac{1}{2}(-\csc x \cot x + \ln|\csc x - \cot x| + C)$$

26. $\int \sin x \sin 4x dx$

let $u = \sin x, du = \cos x dx, dv = \sin 4x dx, v = -\frac{1}{4}\cos 4x$

$$\int \sin x \sin 4x dx = -\frac{1}{4}\sin x \cos 4x + \frac{1}{4}\int \cos x \cos 4x dx$$

let $u' = \cos x, du' = -\sin x dx, dv' = \cos 4x dx, v = \frac{1}{4}\sin 4x$

$$\int \sin x \sin 4x dx = -\frac{1}{4}\sin x \cos 4x + \frac{1}{4}\left(\frac{1}{4}\cos x \sin 4x + \frac{1}{4}\int \sin x \sin 4x dx\right)$$

$$\int \sin x \sin 4x dx = -\frac{1}{4}\sin x \cos 4x + \frac{1}{16}\cos x \sin 4x + \frac{1}{16}\int \sin x \sin 4x dx$$

$$\int \sin x \sin 4x dx - \frac{1}{16}\int \sin x \sin 4x dx = -\frac{1}{4}\sin x \cos 4x + \frac{1}{16}\cos x \sin 4x$$

$$\frac{15}{16}\int \sin x \sin 4x dx = -\frac{1}{4}\sin x \cos 4x + \frac{1}{16}\cos x \sin 4x + C$$

$$\frac{15}{16}\int \sin x \sin 4x dx = \frac{1}{16}(\cos x \sin 4x - 4\sin x \cos 4x) + C$$

$$\int \sin x \sin 4x dx = \frac{1}{15}(\cos x \sin 4x - 4\sin x \cos 4x) + C$$

27. $\int y \cos y \sin^2 y dy$

let u = y, du = dy, dv = cos y sin^2 ydy, $v = \frac{1}{3}\sin^3 y$

$$\int y\cos y\sin^2 y dy = \frac{1}{3}y\sin^3 y - \frac{1}{3}\int \sin^3 y dy$$

$$= \frac{1}{3}y\sin^3 y - \frac{1}{3}\int \sin^2 y\sin y dy$$

$$= \frac{1}{3}y\sin^3 y - \frac{1}{3}\int (1-\cos^2 y)\sin y dy$$

$$= \frac{1}{3}y\sin^3 y - \frac{1}{3}\int \sin y dy + \frac{1}{3}\int \cos^2 y\sin y dy$$

$$= \frac{1}{3}y\sin^3 y + \frac{1}{3}\cos y - \frac{1}{9}\cos^3 y + C$$

28. $\int \ln^2 x dx$

let u = ln^2 x, $du = 2\ln x\frac{dx}{x}, dv = dx, v = x$

$$\int \ln^2 x dx = x\ln^2 x - 2\int \ln x dx$$

$$u' = \ln x, du' = \frac{dx}{x}, dv = dx, v = x$$

$$= x\ln^2 x - 2\left(x\ln x - \int dx\right)$$

$$= x\ln^2 x - 2x\ln x + 2x + C$$

29. $\int \sin(\ln x)dx$

let $u = \sin(\ln x), du = \cos(\ln x)\frac{dx}{x}, dv = dx, v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - \int \cos(\ln x)dx$$

let $u = \cos(\ln x), du = -\sin(\ln x)\frac{dx}{x}, dv = dx, v = x$

$$\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)dx$$

$$2\int \sin(\ln x)dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)dx = \frac{x}{2}[\sin(\ln x) - \cos(\ln x)] + C$$

30. $\int e^{ax}\sin mx dx$

let $u = e^{ax}, du = ae^{ax}dx, dv = \sin mx dx, v = -\frac{1}{m}\cos mx$

$$\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m}\int e^{ax}\cos mx dx$$

$$\text{let } u' = e^{ax}, du' = ae^{ax}dx, dv' = \cos mx dx, v' = \frac{1}{m}\sin mx$$

$$\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m}\left[\frac{1}{m}e^{ax}\sin mx - \frac{a}{m}\int e^{ax}\sin mx dx\right]$$

$$\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m^2}e^{ax}\sin mx - \frac{a^2}{m^2}\int e^{ax}\sin mx dx$$

$$\int e^{ax}\sin mx dx + \frac{a^2}{m^2}\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m^2}e^{ax}\sin mx$$

$$\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m^2}e^{ax}\sin mx - \frac{a^2}{m^2}\int e^{ax}\sin mx dx$$

$$\int e^{ax}\sin mx dx + \frac{a^2}{m^2}\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m^2}e^{ax}\sin mx$$

$$\frac{m^2 + a^2}{m^2}\int e^{ax}\sin mx dx = -\frac{1}{m}e^{ax}\cos mx + \frac{a}{m^2}e^{ax}\sin mx$$

$$\int e^{ax}\sin mx dx = \frac{m^2}{m^2 + a^2}\left(\frac{e^{ax}}{m^2}(a\sin ms - m\cos mx)\right) + C$$

$$\int e^{ax}\sin mx dx = \frac{e^{2x}}{m^2 + a^2}((a\sin ms - m\cos mx)) + C$$

Integration Involving Partial Fractions

1. Resolve $\frac{3x+7}{(x-1)(x-2)}$ into partial fractions, then integrate.

Assume $\frac{3x+7}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$ where A and B are constants to be determined.

$$\frac{3x+7}{(x-1)(x-2)} = \frac{A(x-2) + B(x-1)}{(x-1)(x-2)}$$

Equating numerators, we have

$$3x + 7 = A(x - 2) + B(x - 1)$$

Equating coefficients of x and the constants on each side, we have

3 = A + B
and 7 = -2A – B
giving A = -10 and B = 13.

Another way of solving for A and B

Put x = 1 then $3(1) + 7 = A(1 - 2) + 0$
$A = -10$

Put x = 2 then $3(2) + 7 = 0 + B(2 - 1)$
$B = 13$

Thus $\dfrac{3x+7}{(x-1)(x-2)} = \dfrac{13}{x-2} - \dfrac{10}{x-1}$

Hence, $\displaystyle\int \frac{3x+7}{(x-1)(x-2)}dx = 13\int\frac{dx}{x-2} - 10\int\frac{dx}{x-1}$

$$= 13\ln(x-2) - 10\ln(x-1) + \ln C$$
$$= \ln(x-2)^{13} - \ln(x-1)^{10} + \ln C$$
$$= \ln\frac{(x-2)^{13}C}{(x-1)^{10}}$$

2. Resolve $\dfrac{3x^2 - x + 4}{(x-1)^3}$ into partial fraction then integrate.

Let $\dfrac{3x^2 - x + 4}{(x-1)^3} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$

Putting the right side over the common denominator $(x - 1)^3$, we have

$$\frac{3x^2 - x + 4}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

Equating numerators,

$3x^2 - x + 4 = A(x - 1)^2 + B(x - 1) + C$

Let x = 1 $\qquad 3 - 1 + 4 = C$
$C = 6$

Equating coefficient of x^2: $3 = A$

Equating coefficient of x : $-1 = -2A + B$
$B = -1 + 2A = 5$

Thus, $\frac{3x^2-x+4}{(x-1)^3}=\frac{3}{x-1}+\frac{5}{(x-1)^2}+\frac{6}{(x-1)^3}$

Hence, $\int\frac{3x^2-x+4}{(x-1)^3}$ dx= $3\int\frac{dx}{x-1}+5\int\frac{dx}{(x-1)^2}+6\int\frac{dx}{(x-1)^3}$

$$= 3\ln(x-1)-\frac{5}{x-1}-\frac{3}{(x-1)^2}+C$$

3. Evaluate $\int\frac{-2x+4}{(x^2+1)(x-1)^2}dx$

Let $\frac{-2x+4}{(x^2+1)(x-1)^2}=\frac{Ax+B}{x^2+1}+\frac{C}{x-1}+\frac{D}{(x-1)^2}$

Then,

$$-2x + 4 = (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1)$$

$$= (A + C)x^3 + (-2A + B - C + D)x^2 + (A - 2B + C)x + (B - C + D)$$

Equating these coefficients

Coefficient of x^3 : $0 = A + C$

Coefficient of x^2 : $0 = -2A + B - C + D$

Coefficient of x : $-2 = A - 2B + C$

Coefficient of x^o: $4 = B - C + D$

If we subtract the fourth equation from the second,

$-4 = 2A$, $A = 2$

Then form the first equation,

$C = -A = -2$

Knowing A and C, from the third equation,

$B = 1$

Finally, from the fourth equation, we have,

$D = 4 - B + C = 1$

Hence,

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx = \int \frac{2x+1}{x^2+1}dx - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= \int \frac{2xdx}{x^2+1} + \int \frac{dx}{x^2+1} - 2\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$$

$$= \ln(x^2+1) + ArcTanx - 2\ln(x-1) - \frac{1}{x-1} + C$$

$$= ArcTanx + \ln\frac{x^2+1}{(x-1)^2} - \frac{1}{x-1} + C$$

4. Evaluate $\int \frac{x^2+4}{x-1}dx$

$\frac{x^2+4}{x-1}$ is not a proper fraction and therefore the numerator must be divided by the denominator until the degree of the remainder is less than that of the denominator.

$$\frac{x^2+4}{x-1} = x+1+\frac{5}{x-1}$$

$$\int \frac{x^2+4}{x-1}dx = \int\left(x+1\frac{5}{x-1}\right)dx$$

$$= \int xdx + \int dx + 5\int \frac{dx}{x-1}$$

$$= \frac{x^2}{2} + x + 5\ln|x-1| + C$$

Integrals involving $ax^2 + bx + c$

1. Evaluate $\int \frac{dx}{x^2+4x+8}$

$$\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{x^2+4x+4+4}$$

$$= \int \frac{dx}{\left(x^2+4x+4\right)+4}$$

$$= \int \frac{dx}{(x+2)^2+4}; u = x+2, a = 2$$

$$= \frac{1}{2}\arctan\frac{x+2}{2} + C$$

2. Evaluate $\int \frac{dx}{3x^2+2x+4}$

$$\int \frac{dx}{3x^2+2x+4} = \int \frac{dx}{3\left(x^2+\frac{2}{3}x\right)+4}$$

$$= \int \frac{dx}{3\left(x^2+\frac{2}{3}x+\frac{1}{9}\right)+4-\frac{1}{3}}$$

$$= \frac{1}{3}\int \frac{dx}{\left(x+\frac{1}{3}\right)^2+\frac{11}{3}}; u = x+\frac{1}{3}, a = \sqrt{\frac{11}{3}} = \frac{\sqrt{33}}{3}$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{\sqrt{33}}{3}}\arctan\frac{x+\frac{1}{3}}{\frac{\sqrt{33}}{3}} + C$$

$$= \frac{1}{\sqrt{33}}\arctan\frac{3\left(x+\frac{1}{3}\right)}{\sqrt{33}} + C$$

$$= \frac{1}{\sqrt{33}}\arctan\frac{3x+1}{\sqrt{33}} + C$$

3. Evaluate $\int \frac{dx}{\sqrt{2x-x^2}}$

$$\int \frac{dx}{\sqrt{2x-x^2}} = \int \frac{dx}{\sqrt{-\left(x^2-2x\right)}}$$

$$= \int \frac{dx}{\sqrt{-\left(x^2-2x+1\right)+1}}$$

$$= \int \frac{dx}{\sqrt{1-(x-1)^2}}; u = x-1; a = 1$$

$$= \arcsin(x-1)+C$$

4. $\int \frac{dx}{4x^2+4x+2}$

$$4x^2+4x+2 = 4\left(x^2+x\right)+2$$

$$= 4\left(x^2+x+\frac{1}{4}\right)+(2-1)$$

$$= 4\left(x+\frac{1}{2}\right)^2+1$$

let $u = x+\frac{1}{2}, du = dx$

$$\int \frac{dx}{4x^2+4x+2} = \int \frac{du}{4u^2+1} = \frac{1}{2}\int \frac{du}{u^2+\frac{1}{4}}$$

$$= \frac{1}{4}\frac{1}{\frac{1}{2}}\tan^{-1}\frac{u}{\frac{1}{2}}+C = \frac{1}{2}\tan^{-1}2u+C$$

$$= \frac{1}{2}\tan^{-1}2\left(x+\frac{1}{2}\right)+C = \frac{1}{2}\tan^{-1}(2x+1)+C$$

Integrals involving $\sqrt{ax+b}, \sqrt{x^2+a^2}, \sqrt[n]{x}$

1. Evaluate $\int \frac{xdx}{\sqrt{x+4}}$

Let u = $\sqrt{x+4}$

$u^2 = x + 4$ or $x = u^2 - 4$

dx = 2udy

Substituting the given integrand

$$\int \frac{xdx}{x+4} = \int \frac{(u^2-4)2udu}{u}$$

$$= 2\int (u^2-4)du$$

$$= \frac{2u^3}{3} - 8u + C$$

$$= \frac{2u}{3}(u^2-12) + C$$

Replacing u by $\sqrt{x+4}$

$$= \frac{2}{3}\sqrt{x+4}(x+4-12) + C$$

$$= \frac{2}{3}\sqrt{x+4}(x-8) + C$$

2. Evaluate $\int \frac{(x^3+4x)dx}{\sqrt{x^2-6}}$

Let u = $\sqrt{x^2-6}$

$u^2 = x^2 - 6$ or $x^2 = u^2 + 6$

2xdx = 2udu or xdx = udu

$$\int \frac{(x^3+4x)dx}{\sqrt{x^2-6}} = \int \frac{(x^2+4)xdx}{\sqrt{x^2-6}}$$

$$= \int \frac{[(u^2+6)+4]udu}{u}$$

$$= \int (u^2+10)du$$

$$= \frac{u^3}{3} + 10u + C$$

$$= \frac{u}{3}(u^2+30) + C$$

Replacing u by $\sqrt{x^2-6}$

$$= \frac{\sqrt{x^2-6}}{3}\left(x^2-6+30\right)+C$$

$$= \frac{\sqrt{x^2-6}}{3}\left(x^2+24\right)+C$$

3. Evaluate $\int \frac{(x+2)dx}{\sqrt[4]{(2x-3)^3}}$

$$u = \sqrt[4]{2x-3}$$

$$u^4 = 2x-3$$

Let ***or***

$$x = \frac{u^4+3}{2}$$

$$dx = 4u^3 du$$

$$\int \frac{(x+2)dx}{\sqrt[4]{(2x-3)^3}} = \int \frac{\left(\frac{u^4+3}{2}+2\right)2u^3 du}{u^3}$$

$$= \int \frac{\left(\frac{u^4+3+4}{2}\right)2u^3 du}{u^3}$$

$$= \int \left(u^4+7\right)du$$

$$= \frac{u^5}{5}+7u+C$$

$$= \frac{u}{5}\left(u^4+35\right)+c$$

Replacing u by $\sqrt[4]{2x-3}$,

$$= \frac{\sqrt[4]{2x-3}}{5}\left(2x-3+35\right)+C$$

$$= \frac{\sqrt[4]{2x-3}}{5}\left(2x+32\right)+C$$

4. Evaluate $\int \frac{\sqrt{x}dx}{2+\sqrt[4]{x}}$

$$u = \sqrt[4]{x},$$

Let $u^4 = x,$

$$dx = 4u^3 du$$

$$\int \frac{\sqrt{x}dx}{2+\sqrt[4]{x}} = \int \frac{u^2 \cdot 4u^3 du}{2+u}$$

$$= 4\int \frac{u^5 du}{u+2}$$

$$= 4\int \left(u^4 - u^3 + u^2 - u + 1 - \frac{1}{u+2} \right) du$$

$$= 4\left[\frac{u^5}{5} - \frac{u^4}{4} + \frac{u^3}{3} - \frac{u^2}{2} + u - \ln|u+2| + C \right]$$

Replacing u by $x^{1/4}$,

$$= \frac{4}{5}x^{5/4} - x + \frac{4}{3}x^{3/4} - 2x^{1/2} + 4x^{1/4} - \ln\left|x^{1/4} + 2\right| + C$$

Algebraic Substitution

1. $\int \frac{y^3 dy}{\sqrt{y^2-7}}$

Solution

Let $u = \sqrt{y^2-7}, u^2 = y^2 - 7, u^2 + 7 = y^2, 2udu = 2ydy$

$$\int \frac{y^3 dy}{\sqrt{y^2-7}} = \int \frac{y^2 ydy}{\sqrt{y^2-7}} = \int \frac{(u^2+7)udu}{u} = \int (u^2+7)du$$

$$= \int u^2 du + 7\int du = \frac{u^3}{3} + 7u + C$$

$$= \frac{\left(\sqrt{y^2-7}\right)^3}{3} + 7\sqrt{y^2-7} + C$$

2. $\int \sin\sqrt{x}dx$

Solution

Let $y = \sqrt{x}, y^2 = x, 2ydy = dy$

$$\int \sin\sqrt{y}dy = \int \sin y(2ydy) = 2\int y\sin ydy$$

let u = y, du = dy, dv = sinydy, v = -cos y

$$\int \sin\sqrt{x}dx = 2(-y\cos y + \int \cos ydy) = 2(-y\cos y + \sin y) + C$$

$$= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C$$

3. $\int \frac{x^5}{\sqrt{1-x^2}}dx$

Solution

Let $\sqrt{1-x^2} = u, 1 - x^2 = u^2, x^2 = 1 - u^2, 2xdx = -2udu$

$$\int \frac{x^5}{\sqrt{1-x^2}}dx = \int \frac{(x^2)^2 xdx}{\sqrt{1-x^2}} = -\int \frac{(1-u^2)^2 udu}{u}$$

$$= -\int (1 - 2u^2 + u^4)du$$

$$= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$$

$$= -\sqrt{1-x^2} + \frac{2}{3}\left(\sqrt{1-x^2}\right)^3 - \frac{1}{5}\left(\sqrt{1-x^2}\right)^5 + C$$

4. $\int \frac{x}{\sqrt{1-x^2}}dx$

Solution

Let $u = \sqrt{1-x^2}, u^2 = 1-x^2, 2udu = -2xdx$

$$\int \frac{x}{\sqrt{1-x^2}}dx = -\int \frac{udu}{u} = -\int du$$
$$= -u + C = -\sqrt{1-x^2} + C$$

5. $\int \frac{x^3dx}{\sqrt{1-x^2}}$

Solution

Let $\sqrt{1-x^2} = u, 1-x^2 = u^2, x^2 = 1-u^2, 2xdx = -2udu$

$$\int \frac{x^3dx}{\sqrt{1-x^2}} = \int \frac{(x^2)xdx}{\sqrt{1-x^2}} = -\int \frac{(1-u^2)udu}{u} = -\int du + \int u^2du$$
$$= -u + \frac{1}{3}u^3 = -\sqrt{1-x^2} + \frac{1}{3}\left(\sqrt{1-x^2}\right)^3 + C$$

6. $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

Solution

Let $x = u^6 u = x^{1/6}, dx = 6u^5du, u^3 = x^{1/2}, u^2 = x^{1/3}$

$$\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = \int \frac{6u^5du}{u^3+u^2} = 6\int \frac{u^5du}{u^2(u+1)} = 6\int \frac{u^3du}{u+1}$$
$$= 6\int\left(u^2 - u - \frac{1}{u+1}\right)du = 2u^3 - 3u^2 - 6\ln(u+1) + C$$
$$= 2x^{1/2} - 3x^{1/3} - 6\ln\left(x^{1/6}+1\right) + C$$

7. $\int \cos\sqrt{y}dy$

Solution

Let $x = \sqrt{y}, x^2 = y, 2xdx = dy$

$$\int \cos\sqrt{y}dy = 2\int x\cos x dx$$

Let $u = x, du = dx, dv = \cos x dx, v = \sin x$

$$\begin{aligned}\int \cos\sqrt{y}dy &= 2\int x\cos x dx = 2\left[x\sin x - \int \sin x dx\right] \\ &= 2x\sin x + 2\cos x + C \\ &= 2\sqrt{y}\sin\sqrt{y} + 2\cos\sqrt{y} + C\end{aligned}$$

8. $\int \frac{dx}{x^2\left(a^2 + x^2\right)}$

Solution

Let $u = \frac{a}{x}, du = \frac{-adx}{x^2}, x = \frac{a}{u}$

$$\begin{aligned}\int \frac{dx}{x^2\left(a^2 + x^2\right)} &= -\frac{1}{a}\int \frac{1}{a^2 + x^2}\left(\frac{adx}{x^2}\right) = -\frac{1}{a}\int \frac{1}{a^2 + \left(\frac{a}{u}\right)^2}du \\ &= -\frac{1}{a}\int \frac{1}{a^2 + \frac{a^2}{u^2}}du = -\frac{1}{a}\int \frac{1}{a^2\left(1 + \frac{1}{u^2}\right)}du \\ &= -\frac{1}{a^3}\int \frac{1}{\left(\frac{u^2 + 1}{u^2}\right)}du = -\frac{1}{a^3}\int \frac{u^2 du}{u^2 + 1} \\ &= -\frac{1}{a^3}\int\left(1 - \frac{1}{u^2 + 1}\right)du = -\frac{1}{a^3}\left(u - \tan^{-1}u\right) + C \\ &= -\frac{1}{a^3}\left(\frac{a}{x} - \tan^{-1}\frac{a}{x}\right) + C \\ &= \frac{1}{a^3}\tan\frac{a}{x} - \frac{1}{a^2 x} + C\end{aligned}$$

9. $\int \sqrt{1 + \sqrt{x}}dx$

Solution

.Let $y=\sqrt{1+\sqrt{x}}, y^2=1+\sqrt{x}, \left(y^2-1\right)^2=x, 2\left(y^2-1\right)2ydy=dx$

$$\int\sqrt{1+\sqrt{x}}\,dx=4\int y\left(y^2-1\right)ydy=4\int\left(y^4-y^2\right)dy$$
$$=\frac{4}{5}y^5-\frac{4}{3}y^3+C$$
$$=\frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5-\frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3+C$$

10. $\int\sqrt{e^x-9}dx$

Solution

Let $u=\sqrt{e^x-9}, u^2=e^x-9, u^2+9=e^x, 2udu=e^x dx, dx=\dfrac{2udu}{e^x}$

$$\int\sqrt{e^x-9}dx=2\int\frac{u^2du}{u^2+9}=2\left[\int\left(1-\frac{9}{u^2+9}\right)du\right]=2u-\frac{18}{3}\tan^{-1}\frac{u}{3}+C$$
$$=2\sqrt{e^x-9}-6\tan^{-1}\frac{\sqrt{e^x-9}}{3}+C$$

Trigonometric Substitution

When $\sqrt{a^2-u^2}$ occurs, let $u=a\sin\theta$
When $\sqrt{a^2+u^2}$ occurs, let $u=a\tan\theta$
When $\sqrt{u^2-a^2}$ occurs, let $u=a\sec\theta$

1. $\int\dfrac{\sqrt{a^2-x^2}}{x^2}dx$

Solution

Let $x=a\sin\theta, dx=a\cos\theta, \theta=\sin^{-1}\dfrac{x}{a}$

$$\int\frac{\sqrt{a^2-x^2}}{x^2}dx=\int\frac{\sqrt{a^2-a^2\sin^2\theta}}{a^2\sin^2\theta}a\cos\theta d\theta=\int\frac{a^2\cos^2\theta d\theta}{a^2\sin^2\theta}$$
$$=\int\cot^2\theta d\theta=\int\left(\csc^2\theta-1\right)d\theta$$
$$=-\cot\theta-\theta+C$$
$$=-\frac{\sqrt{a^2-x^2}}{x}-\sin^{-1}\frac{x}{a}+C$$

2. $\int \frac{dx}{x\sqrt{x^4-4}}$

Solution

Let $x^2 = 2\sec\theta, x = \sqrt{2\sec\theta}, dx = \frac{\sec\theta\tan\theta d\theta}{\sqrt{2\sec\theta}}$

$$\int \frac{dx}{x\sqrt{x^4-4}} = \int \frac{\frac{\sec\theta\tan\theta}{\sqrt{2\sec\theta}}d\theta}{\sqrt{2\sec\theta}\sqrt{4\sec^2\theta-4}}$$
$$= \frac{1}{4}\int d\theta = \frac{1}{4}\theta + C$$
$$= \frac{1}{4}\sec^{-1}\frac{x^2}{2} + C$$

3. $\int \frac{dx}{(x^2+2x+2)^{3/2}}$

Solution

$$\int \frac{dx}{(x^2+2x+2)^{3/2}} = \int \frac{dx}{[(x^2+2x+1)+1]^{3/2}} = \int \frac{dx}{[(x+1)^2+1]^{3/2}}$$

Let $x+1 = \tan\theta, dx = \sec^2\theta d\theta$

$$\int \frac{dx}{(x^2+2x+2)^{3/2}} = \int \frac{\sec^2\theta d\theta}{[\tan^2\theta+1]^{3/2}} = \int \frac{\sec^2\theta d\theta}{\sec^3\theta}$$
$$= \int \frac{d\theta}{\sec\theta} = \int \cos\theta d\theta = \sin\theta + C$$
$$= \frac{x+1}{\sqrt{x^2+2x+2}} + C$$

4. $\int \sqrt{r^2-x^2}dx$

Solution

Let $x = r\sin\theta, \theta = \sin^{-1}\frac{x}{r}, dx = r\cos\theta d\theta$

$$\int\sqrt{r^2-x^2}\,dx = r^2\int\cos^2\theta d\theta = r^2\int\frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= \frac{r^2}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+C$$

$$= \frac{r^2\theta}{2}+\frac{r^2\sin\theta\cos\theta}{2}+C$$

$$= \frac{1}{2}\left(r^2\sin^{-1}\frac{x}{r}+x\sqrt{r^2-x^2}\right)+C$$

5. $\int\frac{x^2}{\sqrt{a^2-x^2}}dx$

Solution

Let $x = a\sin\theta, dx = a\cos\theta d\theta, \sqrt{a^2-x^2} = a\cos\theta$

$$\int\frac{x^2}{\sqrt{a^2-x^2}}dx = \int\frac{a^2\sin^2\theta}{a\cos\theta}a\cos\theta d\theta = a^2\int\sin^2\theta d\theta$$

$$= \frac{a^2}{2}\int(1-\cos 2\theta)d\theta$$

$$= \frac{a^2}{2}\left(\theta-\frac{1}{2}\sin 2\theta\right)+C$$

$$= \frac{a^2}{2}\theta-\frac{a^2}{2}\sin\theta\cos\theta+C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{a^2}{2}\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}+C$$

$$= \frac{a^2}{2}\sin^{-1}\frac{x}{a}-\frac{x\sqrt{a^2-x^2}}{2}+C$$

6. $\int\frac{dx}{\sqrt{a^2+x^2}}$

Solution

Let $x = a\tan\theta, dx = a\sec^2\theta d\theta, \sqrt{a^2+x^2} = a\sec\theta$

$$\int\frac{dx}{\sqrt{a^2+x^2}} = \int\frac{a\sec^2\theta d\theta}{\sec\theta} = \int\sec\theta d\theta$$

$$= \ln|\sec u+\tan u|+C$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}}{a}+\frac{x}{a}\right|+C$$

7. $\int \frac{dx}{\left(\sqrt{a^2+x^2}\right)^3}$

Solution

Let $x = a\tan\theta, dx = a\sec^2\theta d\theta, \sqrt{a^2+x^2} = a\sec\theta$

$$\int \frac{dx}{\left(\sqrt{a^2+x^2}\right)^3} = \int \frac{a\sec^2\theta d\theta}{a^3\sec^3\theta} = \frac{1}{a^2}\int \frac{d\theta}{\sec\theta} = \frac{1}{a^2}\int \cos\theta d\theta$$

$$= \frac{1}{a^2}\sin\theta + C$$

$$= \frac{x}{a^2\sqrt{a^2+x^2}} + C$$

8. $\int \frac{x^2 dx}{\left(9-x^2\right)^{3/2}}$

Solution

Let $x = 3\sin\theta, dx = 3\cos\theta d\theta, \left(\sqrt{9-x^2}\right)^3 = 3^3\cos^3\theta$

$$\int \frac{x^2 dx}{\left(\sqrt{9-x^2}\right)^3} = \int \frac{3^2\sin^2\theta 3\cos\theta d\theta}{3^3\cos^3\theta} = \int \frac{\sin^2\theta}{\cos^2\theta} d\theta$$

$$= \int \tan^2\theta d\theta = \int \left(\sec^2\theta - 1\right) d\theta$$

$$= \int \sec^2\theta d\theta - \int d\theta$$

$$= \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{9-x^2}} + \sin^{-1}\frac{x}{3} + C$$

9. $\int \frac{dx}{x\sqrt{a^2-x^2}}$

Solution

Let $x = a\sin\theta, dx = a\cos\theta d\theta, \sqrt{a^2 - x^2} = a\cos\theta, \frac{x}{a} = \sin\theta, \frac{\sqrt{a^2 - x^2}}{a} = \cos\theta$

$$\int \frac{dx}{x\sqrt{a^2 - x^2}} = \int \frac{a\cos\theta d\theta}{a\sin\theta a\cos\theta} = \frac{1}{a}\int \frac{d\theta}{\sin\theta} = \frac{1}{a}\int \csc\theta d\theta$$

$$= \frac{1}{a}\ln\left|\csc\theta - \cot\theta\right|$$

$$= \frac{1}{a^2}\ln\left|\frac{1}{x} - \frac{1}{\sqrt{a^2 - x^2}}\right| + C$$

10. $\int \frac{dx}{x\sqrt{x^2 - a^2}}$

Solution

Let $x = a\sec\theta, dx = a\sec\theta\tan\theta d\theta, \sqrt{x^2 - a^2} = a\tan\theta, \frac{x}{a} = \sec\theta$

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \int \frac{a\sec\theta\tan\theta d\theta}{a\sec\theta a\tan\theta} = \frac{1}{a}\int d\theta$$

$$= \frac{1}{2}\theta + C$$

$$= \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

Definite Integrals

1. $\int_1^2 (2x + 1)dx$

Solution.

$$\int_1^2 (2x + 1)dx = (x^2 + x)\Big]_1^2$$

$$= (2^2 + 2) - (1^2 + 1)$$

$$= 6 - 2 = 4$$

2. $\int_0^{\pi/2} \cos x dx$

Solution.

$$\int_0^{\pi/2} \cos x\,dx = \sin x \Big]_0^{\pi/2}$$

$$= \sin\frac{\pi}{2} - \sin 0$$

$$= 1$$

3. $\displaystyle\int_1^2 8e^{2x}\,dx$

Solution.

$$\int_1^2 8e^{2x}\,dx = 8\int_1^2 e^{2x}\,dx$$

$$= 8\left[\frac{1}{2}e^{2x}\right]_1^2$$

$$= 4\left(e^4 - e^2\right)$$

4. $\displaystyle\int_1^2 \left(4 + \frac{8}{x^2}\right)dx$

Solution.

$$\int_1^2 \left(4 + \frac{8}{x^2}\right)dx = 4\int^2 dx + 8\int^2 \frac{dx}{x^2}$$

$$= 4x\Big]_1^2 + 8\left(-\frac{1}{x}\right)_1^2$$

$$= 4(2-1) + 8\left(-\frac{1}{2} + 1\right)$$

$$= 8$$

5. $\displaystyle\int^0 \left(5 - 6x^2\right)dx$

Solution

$$\int_1^0 (5-6x^2)dx = \int_0^1 (6x^2-5)dx$$
$$= (2x^3-5x)]_0^1$$
$$= 2-5$$
$$= -3$$

6. $\int_0^4 (x^2-\sqrt{x})dx$

Solution

$$\int_0^4 (x^2-\sqrt{x})dx = 1\int_0 (x^2-x^{1/2})dx$$
$$= \left[\frac{x^3}{3} - \frac{x^{3/2}}{\frac{3}{2}}\right]_0^4 = \frac{64}{3} - \frac{16}{3} = 16$$

7. $\int_{1/2}^1 \cos x dx$

Solution

$$\int_{1/2}^1 \cos x dx = [\sin x]_{1/2}^1 = \sin 1 - \sin\frac{1}{2}$$,(where the angles are in radians)
$$= \sin 57.3^o - \sin 28.65^o$$
$$= 0.8415 - 0.4794$$
$$= 0.362$$

8. $\int_0^2 \frac{dx}{e^x}$

Solution

$$\int_0^2 \frac{dx}{e^x} = \int e^{-x}dx = [-e^{-x}]_0^2 = (-e^{-2}+e^0)$$
$$= 1-0.135$$
$$= 0.865$$

9. $\int^2 \frac{e^x+e^{-x}}{2}dx$

Solution

$$\int_1^2 \frac{e^x + e^{-x}}{2} dx = \frac{1}{2} \int \left(e^x + e^{-x}\right) dx$$

$$= \frac{1}{2} \left[e^x - e^{-x}\right]_1^2$$

$$= \frac{1}{2}\left(e^2 - \frac{1}{e^2}\right) - \frac{1}{2}\left(e - \frac{1}{e}\right)$$

$$= \frac{1}{2}(7.3891 - 0.1353) - \frac{1}{2}(2.7183 - 0.3679)$$

$$= 2.45$$

10. $\int_4^6 \frac{x^2 + 2}{x - 2} dx$

Solution

$$\int_4^6 \frac{x^2 + 2}{x - 2} dx = \int_4^6 \left(x + 2 + \frac{6}{x - 2} \right) dx$$

$$= \left[\frac{x^2}{2} + 2x + 6\ln(x - 2) \right]_4^6$$

$$= 18 + 12 + 6\ln 4 - (8 + 8 + 6\ln 2)$$

$$= 14 + 6\ln 2$$

$$= 18.2$$

11. $\int_1^3 x^3 \ln x dx$

Solution

$$\int_1^3 x^3 \ln x dx = \left[\frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} dx \right]_1^3$$

$$= \left[\frac{x^4 \ln x}{4} - \frac{1}{4} \int x^3 dx \right]_1^3$$

$$= \left[\frac{x^4 \ln x}{4} - \frac{x^4}{16} \right]_1^3$$

$$= \frac{81 \ln 3}{4} - \frac{81}{16} - \left(0 - \frac{1}{16}\right)$$

$$= 17.2$$

12. $\int_1^2 \frac{dx}{4+x^2}$

Solution

$$\int_1^2 \frac{dx}{4+x^2} = \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_1^2$$

$$= \frac{1}{2}\left(\tan^{-1}1 - \tan^{-1}\frac{1}{2}\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0.4637\right)$$

$$= 0.161$$

13. $\int_0^2 2x^2\sqrt{x^3+1}\,dx$

Solution

$$\int_0^2 2x^2\sqrt{x^3+1}\,dx = \frac{2}{3}\int_0^2 \sqrt{x^3+1}\left(3x^2dx\right)$$

$$= \frac{2}{3}\left[\frac{\left(x^3+1\right)^{3/2}}{\frac{3}{2}}\right]_0^2$$

$$= \frac{4}{9}\left[\left(x^3+1\right)^{3/2}\right]_0^2$$

$$= \frac{4}{9}\left[(8+1)^{3/2} - (0+1)^{3/2}\right]$$

$$= \frac{4}{9}(27-1)$$

$$= \frac{104}{9}$$

14. $\int_0^3 x\sqrt{x+1}\,dx$

Solution

Let u = $\sqrt{x+1}$, u^2 = x+1, x = u^2 – 1, dx = 2udu,

$$\int_0^3 x\sqrt{x+1}\,dx = \int_1^2 \left(u^2-1\right)u\left(2udu\right)$$

$$= 2\int_c^d \left(u^4 - u^2\right)du$$

$$= \frac{2}{5}u^5 - \frac{2}{3}u^3 \Big]_c^d$$

$$= \left[\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2}\right]_0^3$$

$$= \frac{2}{5}(4)^{5/2} - \frac{2}{3}(4)^{3/2} - \frac{2}{5}(1)^{5/2} + \frac{2}{3}(1)^{3/2}$$

$$= \frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3}$$

$$= \frac{116}{15}$$

15. $\int_0^{\pi/2} \sin^3 x \cos x dx$

Solution

Let u = sin x, du = cos x dx, when x = 0, u = 0, when x = $\frac{\pi}{2}$, u = 1

$$\int_0^{\pi/2} \sin^3 x \cos x dx = \int_0^1 u^3 du$$

$$= \left[\frac{u^4}{4}\right]_0^1$$

$$= \frac{1}{4}$$

16. $\int_{-3}^{1} |x+2| dx$

Solution

$$|x+2| = \begin{cases} -x-2, x \le -2 \\ x+2, -2 \le x \end{cases}$$

$$\int_{-3}^{1} |x+2| dx = \int_{-3}^{-2} (-x-2)dx + \int_{-2}^{1} (x+2)dx$$

$$= \left[-\frac{x^2}{2} - 2x\right]_{-3}^{-2} + \left[\frac{x^2}{2} + 2x\right]_{-2}^{4}$$

$$= \left[(-2+4) - \left(-\frac{9}{2} + 6\right)\right] + \left[(8+8) - (2-4)\right]$$

$$= \frac{1}{2} + 18$$

$$= \frac{37}{2}$$

17. $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}}$

Solution

$$\int_{-1}^{1} \frac{dx}{\sqrt{(x+2)(3-x)}} = \int_{-1}^{1} \frac{dx}{\sqrt{6+x-x^2}} = \int_{-1}^{1} \frac{dx}{\sqrt{6-(x^2-x)}}$$

$$= \int_{-1}^{1} \frac{dx}{\sqrt{\frac{25}{4} - \left(x^2 - x + \frac{1}{4}\right)}} = \int_{-1}^{1} \frac{dx}{\sqrt{\frac{25}{4} - \left(x - \frac{1}{2}\right)^2}}$$

$$= \left[\sin^{-1} \frac{x - \frac{1}{2}}{\frac{5}{2}}\right]_{-1}^{1} = \left[\sin^{-1} \frac{2x-1}{5}\right]_{-1}^{1}$$

$$= \sin^{-1} \frac{1}{5} - \sin^{-1}\left(-\frac{3}{5}\right)$$

$$= \sin^{-1} 0.2 + \sin^{-1} 0.6$$

18. $\displaystyle\int_0^{1/\sqrt{2}} \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx$

Solution

let $\sin^{-1} x^2 = u, du = \dfrac{1}{\sqrt{1-(x^2)^2}} 2x dx = \dfrac{2x dx}{\sqrt{1-x^4}}$

$$\int_0^{1/\sqrt{2}} \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx = \frac{1}{2} \int u du = \frac{1}{4} u^2 + C = \frac{1}{4}\left(\sin^{-1} x^2\right)^2 + C$$

$$\int_0^{1/\sqrt{2}} \frac{x \sin^{-1} x^2}{\sqrt{1-x^4}} dx = \left[\frac{1}{4}\left(\sin^{-1} x^2\right)^2\right]_0^{1/\sqrt{2}} = \frac{1}{4}\left(\sin^{-1} \frac{1}{2}\right)^2 = \frac{\pi^2}{144}$$

19. $\int_1^2 \frac{dx}{\left(x^2-2x+4\right)^{3/2}}$

Solution

$$\int_1^2 \frac{dx}{\left(x^2-2x+4\right)^{3/2}} = \int_1^2 \frac{dx}{\left[(x-1)+3\right]^{3/2}}$$

let $x-1=\sqrt{3}\tan u, dx=\sqrt{3}\sec^2 du$,when x = 1, u = $\tan^{-1} 0 = 0$, when x = 2, $u=\tan^{-1}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$

$$\int_1^2 \frac{dx}{\left(x^2-2x+4\right)^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2 udu}{\left[3+3\tan^2 u\right]^{3/2}} = \int_0^{\pi/6} \frac{\sqrt{3}\sec^2 udu}{\left[3\sec^2 u\right]^{3/2}}$$

$$= \int_0^{\pi/6} \frac{1}{\sec u} du = \frac{1}{3}\int_0^{\pi/6} \cos udu$$

$$= \frac{1}{3}\left[\sin u\right]_0^{\pi/6} = \frac{1}{6}$$

20. $\int_e^{e^2} \frac{dx}{x(\ln x)^3}$

Solution

let lnx = y, dy/dx = dy, when x = e, y = 1, when x = e^2, y = 2

$$\int_e^{e^2} \frac{dx}{x(\ln x)^3} = \int_1^2 \frac{dy}{y} = \left[\frac{y^{-2}}{-2}\right]_1^2 = \left[-\frac{1}{2y^2}\right]_1^2$$

$$= -\frac{1}{8}+\frac{1}{2}=\frac{3}{8}$$

21. $\int_0^1 x\ln(x+3)dx$

Solution

Let $u=\ln(x+3), du=\frac{dx}{x+3}, dv=xdx, v=\frac{x^2}{2}$

$$\int_0^1 x\ln(x+3)dx = \frac{x^2}{2}\ln(x+3)-\frac{1}{2}\int\frac{x^2}{x+3}dx$$

$$= \frac{x^2}{2}\ln(x+3) - \frac{1}{2}\int\left(x - 3 + \frac{9}{x+3}\right)dx$$

$$= \left[\frac{x^2}{2}\ln(x+3) - \frac{x^2}{4} + \frac{3}{2}x - \frac{9}{2}\ln(x+3)\right]_0^1$$

$$= \frac{5}{4} - 4\ln 4 + \frac{9}{2}\ln 3$$

22. $\int_0^5 \frac{3(x-1)}{(3x+1)^{3/2}}dx$

Solution

Let $3x+1 = u^{3/2}, x = \frac{1}{3}\left(u^{3/2} - 1\right), dx = \frac{2}{9}u^{-1/3}du$

$$\int_0^5 \frac{3(x-1)}{(3x+1)^{3/2}}dx = \frac{2}{9}\int\left(u^{-2/3} - 4u^{-4/3}\right) = \frac{2}{3}\left(u^{1/3} + \frac{4}{u^{1/3}}\right)$$

$$= \frac{2}{3}\left[\sqrt{3x+1} + \frac{4}{\sqrt{3x+1}}\right]_0^5 = 0$$

23. $\int_4^9 \frac{dx}{x - \sqrt{x}}$

Solution

Let $\sqrt{x} = u, x = u^2, dx = 2udu$. When x = 4, u = 2, and when x = 9, u = 3

$$\int_4^9 \frac{dx}{x - \sqrt{x}} = 2\int_2^3 \frac{udu}{u^2 - u} = 2\int \frac{du}{u-1}$$

$$= \left[2\ln|u-1|\right]_2^3 = 2\ln 2 = \ln 4$$

24. $\int^2 \frac{dx}{x^2\sqrt{4-x^2}}$

Solution

Let $x = 2\sin\theta, dx = 2\cos\theta d\theta, \sqrt{4-x^2} = 2\cos\theta$

$$\int^2 \frac{dx}{x^2\sqrt{4-x^2}} = \int\frac{2\cos\theta d\theta}{(2\sin\theta)^2 2\cos\theta} = \frac{1}{4}\int\frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int\csc^2\theta$$

$$= -\frac{1}{4}[\cot\theta] = -\frac{1}{4}\left[\frac{\sqrt{4-x^2}}{x}\right]_1^2$$

$$= \frac{\sqrt{3}}{4}$$

25. $\int_0^1 \frac{x}{(x+1)^2}dx$

Solution

Let u = x, du = dx, $dv = \frac{dx}{(x+1)^2}, v = \frac{-1}{x+1}$

$$\int_0^1 \frac{x}{(x+1)^2}dx = -\frac{x}{x+1} - \int_0^1 \frac{-1}{x+1}dx$$

$$= \left[-\frac{x}{x+1} + \ln|x+1|\right]_0^1$$

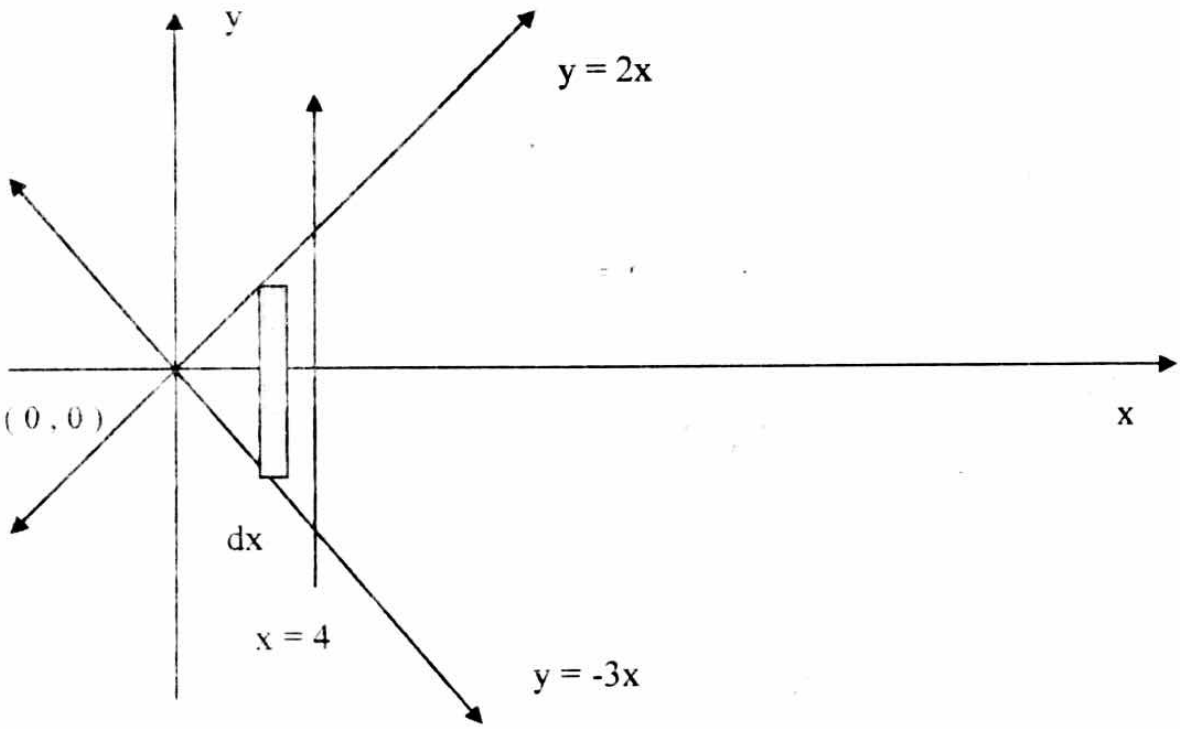
$$= \ln 2 + \frac{1}{2}$$

Geometric Applications

Areas

1. Find the area of the region bounded by $y = 2x$, $y = -3x$, and $x = 4$.

Solution :



Based on the figure above ,

$f(x) \rightarrow y = 2x$
$g(x) \rightarrow y = -3x$

Limit is from zero to four or [0, 4]

$$A = \int_a^b [\, f(x) - g(x) \,]\, dx$$

$$= \int_0^4 [\, (2x) - (-3x) \,]\, dx$$

$$= \int_0^4 5x \, dx$$

$$= \left. \frac{5x^2}{2} \right]_0^4$$

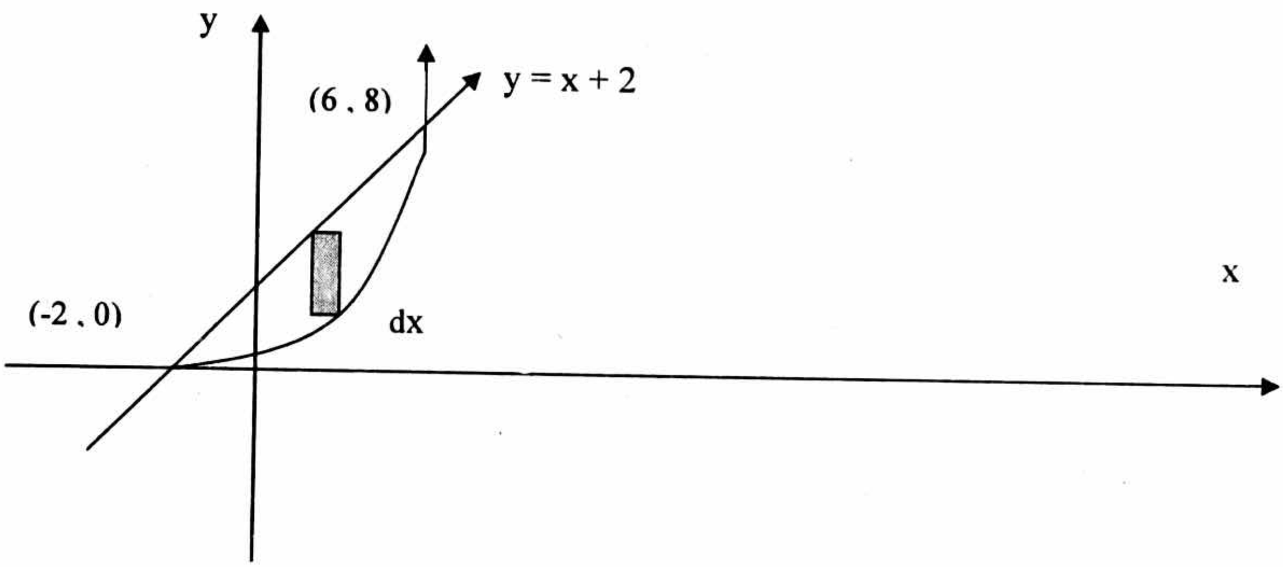
$$= \frac{5(4)^2}{2} - \frac{5(0)^2}{2}$$

$= 5(8)$

$= 40$ sq. units

2. Find the area of the region bounded by $x^2 = 4y + 4$ and $y = x + 2$.

Solution :



Consider : $f(x) \rightarrow y = x + 2$ and $x^2 = 4y + 4 \rightarrow 4y = x^2 - 4$ thus, $g(x) \rightarrow y = \dfrac{x^2 - 4}{4}$ from the figure above the coordinates of the intersections are ,

$$x + 2 = \frac{x^2 - 4}{4}$$

$$4 (x + 2) = x^2 - 4$$

then, $x^2 - 4x - 12 = 0 \rightarrow (x - 6) (x + 2) = 0 \rightarrow x_1 = -2$; $x_2 = 6$, so interval is [-2, 6]

$$A = \int_{-2}^{6} [(x + 2) - \frac{x^2 - 4}{4}] \, dx$$

$$A = \int_{-2}^{6} [x + 2 - \tfrac{1}{4} (x^2 + 4)] \, dx$$

$$A = \int_{-2}^{6} (x + 2 - \tfrac{1}{4} x^2 + 1) \, dx$$

$$A = \tfrac{1}{2} x^2 - 1/12 \, x^3 + 3x]_{-2}^{6}$$

$A = [\ \frac{1}{2}(6)^2 - 1/12\,(6)^3 + 3(6)\ ] - [\ \frac{1}{2}(-2)^2 - 1/12\,(-2)^3 + 3(-2)\ ]$

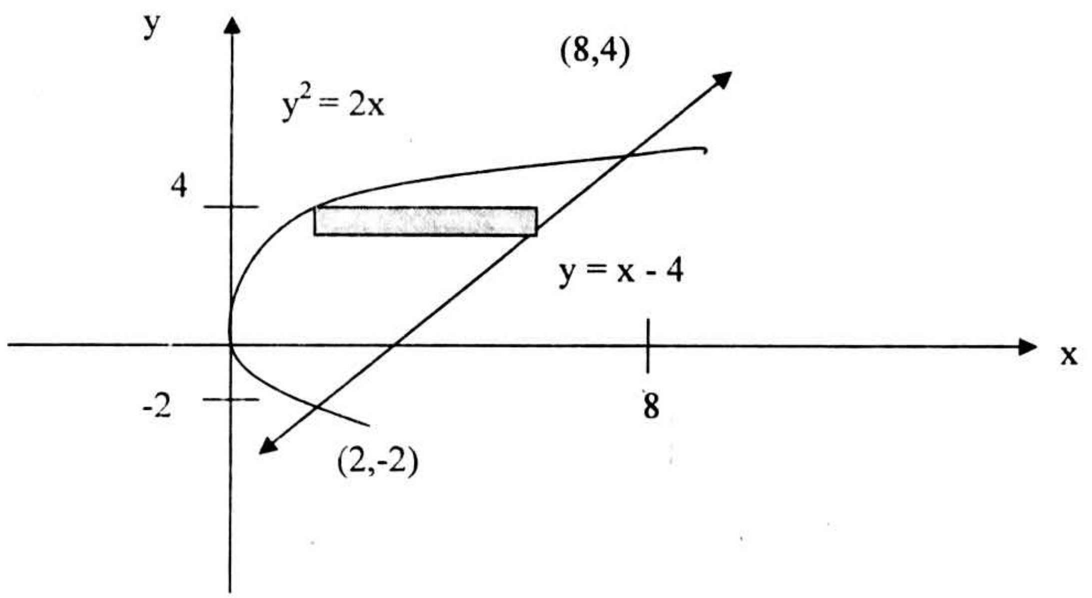
$A = (\ 18 - 18 + 18\ ) - (\ 2 + 2/3 - 6\ )$

$A = 18 - (\ -10/3\ )$

$A = 64/3$ sq. units

3. Find the area bounded by the curves $y^2 = 2x$ and $y = x - 4$.

Solution :



Using the horizontal element, we can consider $y = x - 4 \rightarrow x = y + 4$ as f(x) and $y^2 = 2x \rightarrow x = \frac{1}{2}y^2$ as g(x). Based from above graph, the interval is [-2, 4].

$A = \int_{-2}^{4} [\ (\ y + 4\ ) - (\ \frac{1}{2}y^2\ )\ ]\ dy$

$A = \frac{1}{2}y^2 + 4y - 1/6\,y^3]_{-2}^{4}$

$A = [\ \frac{1}{2}(4)^2 + 4(4) - 1/6\,(4)^3] - [\ \frac{1}{2}(-2)^2 + 4(-2) - 1/6\,(-2)^3\ ]$

$A = (\ 8 + 16 - 64/6\ ) - (\ 2 - 8 + 8/6\ )$

$A = (\ 80/6\ ) - (\ -28/6\ )$

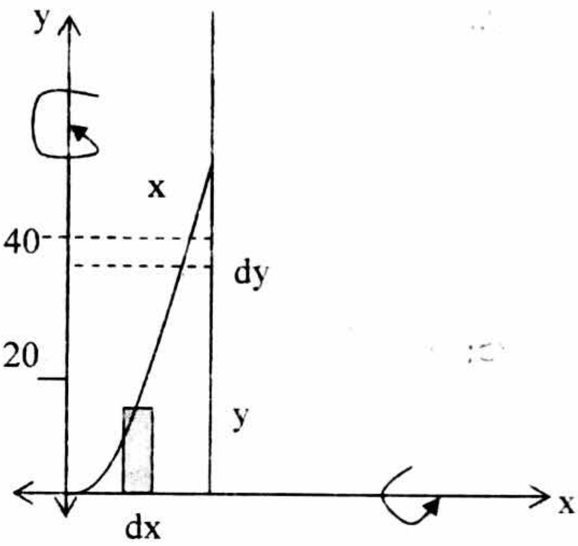
$A = 18$ sq. units

Volumes

Find the volume generated by revolving the given plane area about the given line :

1. $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about x-axis

Solution :



x = 0	x = 5	y = 0
$y = 2(0)^2$	$y = 2(5)^2$	$0 = 2x^2$
$y = 0$	$y = 50$	$0 = x$

Hence, the Point of Intersection : (0, 0) and (5, 50), thus limits are 0 and 5.
Using Circular Disk Formula the Volume :

$$V = \pi \int_a^b r^2 dr$$

$$V = \pi \int_a^b y^2 dx$$

Replacing y in terms of x and simplifying :

$$V = \pi \int_0^5 \left(2x^2\right)^2 dx$$

$$V = \pi \int_0^5 \left(4x^4\right) dx$$

$$V = \pi \left[\frac{4x^5}{5}\right]_0^5$$

$$V = \pi \left[\frac{4(5)^5}{5} - \frac{4(0)^5}{5}\right]$$

$$V = 2500\pi cu.units$$

2. $y = 2x^2$, $y = 0$, $x = 0$, $x = 5$; about y-axis.

Solution :

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

$$V = \pi \int_0^{50} \left[(5)^2 - \left(\sqrt{\frac{y}{2}} \right)^2 \right] dy$$

$$V = \pi \int_0^{50} \left(25 - \tfrac{y}{2} \right) dy$$

$$V = \pi \left[25y - \tfrac{y^2}{4} \right]_0^{50}$$

$$V = \pi \left[25(50) - \tfrac{(50)^2}{4} \right] - \left[25(0) - \tfrac{(0)^2}{4} \right]$$

$$V = 625\pi cu.units$$

Alternative solution:

$$V = 2\pi \int_a^b x \left[f(x) - g(x) \right] dx$$

In which, [f(x) – g(x)] represent the height, then h (height) is the $y = 2x^2$, since $2x^2 - 0$) is $2x^2$, thus

$$V = 2\pi \int_a^b xhdx$$

$$V = 2\pi \int_0^5 x(2x^2)dx$$

$$V = 2\pi \int_0^5 2x^3 dx$$

$$V = 2\pi \left[\tfrac{x^4}{2} \right]_0^5$$

$$V = 2\pi \left[\tfrac{(5)^4}{2} - \tfrac{(0)^4}{2} \right]$$

$$V = \pi(5)^4$$

$$V = 625\pi cu.units$$

3. $x = 9 - y^2$, $x - y - 7 = 0$; about $x = 4$.

Solution :

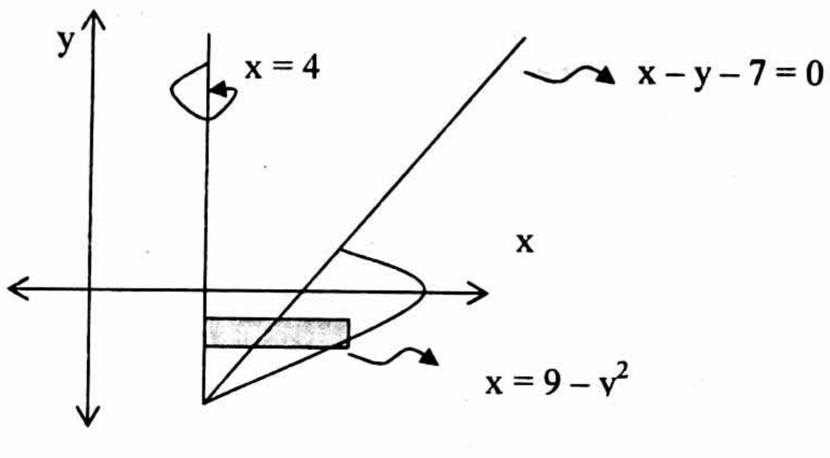
$x = 9 - y^2$ (Eq. 1)
$y^2 = -x + 9$
$y^2 = -(x - 9)$ V(9, 0)

$x - y - 7 = 0$ (Eq. 2)
$x = y + 7$

$y + 7 = 9 - y^2$ $(y + 2)(y - 1) = 0$; $y = -2$ and $y = 1$
$y^2 + y - 2 = 0$

$x = y + 7$ ---- $x = -2 + 7$ ---- $x = 5$ and $x = 1 + 7$ ---- $x = 8$

Hence, the points of intersection are (5, -2) and (8, 1) which serve as the limits of integration. Sketching the graph using Horizontal strip :



Using the formula for Circular Ring :

$$V = \pi \int_a^b \left\{ [f(x)]^2 - [g(x)]^2 \right\} dx$$

Replacing f(x) by (4 – xp) and g(x) by (4 – xl) ; where $xp = 9 - y^2$ and $xl = y + 7$:

$$V = \pi \int_{-2}^{1} [(4-xp)^2 - (4-xl)^2] dy$$

$$V = \pi \int_{-2}^{1} \{[4-(9-y^2)]^2 - [4-(y+7)]^2\} dy$$

$$V = \pi \int_{-2}^{1} [(y^2-5)^2 - (-y-3)^2] dy$$

$$V = \pi \int_{-2}^{1} [(y^4 - 10y^2 + 25) - (y^2 + 6y + 9)] dy$$

$$V = \pi \int_{-2}^{1} (y^4 - 11y^2 - 6y + 16) dy$$

$$V = \pi [\tfrac{y^5}{5} - \tfrac{11y^3}{3} - \tfrac{6y^2}{2} + 16y]_{-2}^{1}$$

$$V = \pi \{[\tfrac{(1)^5}{5} - \tfrac{11(1)^3}{3} - \tfrac{6(1)^2}{2} + 16(1)] - [\tfrac{(-2)^5}{5} - \tfrac{11(-2)^3}{3} - \tfrac{6(-2)^2}{2} + 16(-2)]\}$$

$$V = \pi [(\tfrac{143}{15}) - (\tfrac{-316}{15})]$$

$$V = \pi (\tfrac{459}{15})$$

$$V = \tfrac{153}{5} \pi cu.units$$

4. $y = x^3$, $y = 0$, $x = 2$; about $y = 8$.

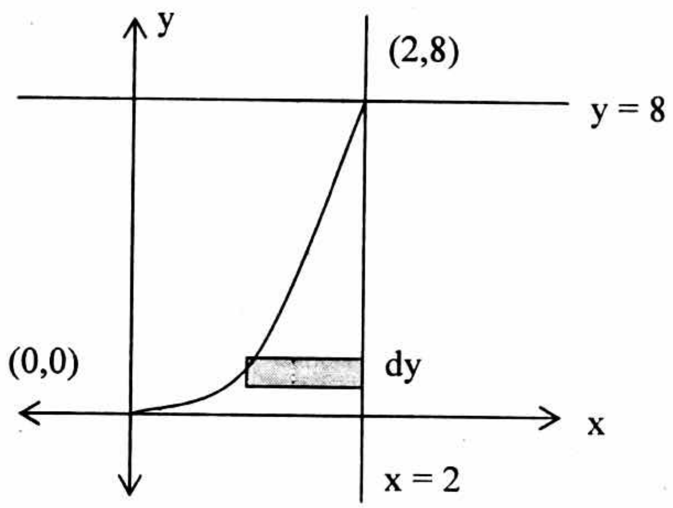
Solution :

Applying the Shell method :

Substituting $y = 0$ and $x = 2$ to $y = x^3$

$y = (2)^3 0 = x^3$
$y = 8$ $\qquad x = 0$ Hence, the points of intersection are (2, 8) and (0, 0).

Sketching the graph :



Utilizing the formula for Shell method :

$$V = 2\pi \int_a^b x[f(x) - g(x)]dx$$

Replacing [f(x) – g(x)] by (8 – y), dx by dy , and x by $(2-\sqrt[3]{y})$: note that $x^3 = y$.

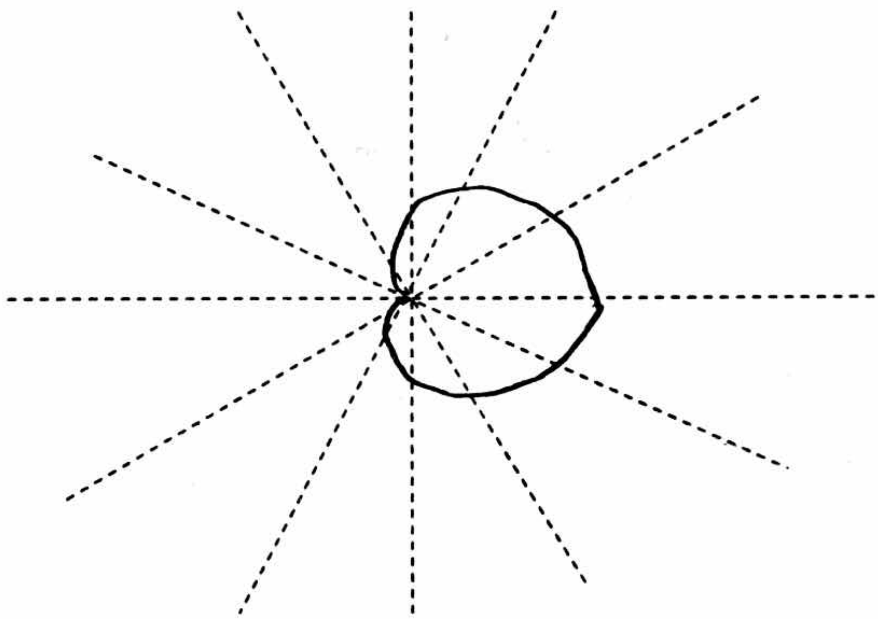
$$V = 2\pi \int_0^8 (2-\sqrt[3]{y})(8-y)dy$$
$$V = 2\pi \int_0^8 (16 - 8y^{\frac{1}{3}} - 2y + y^{\frac{4}{3}})dy$$
$$V = 2\pi[16y - \tfrac{8y^{\frac{4}{3}}}{\frac{4}{3}} - \tfrac{2y^2}{2} + \tfrac{y^{\frac{7}{3}}}{\frac{7}{3}}]_0^8$$
$$V = 2\pi[16y - 6y^{\frac{4}{3}} - y^2 + \tfrac{3}{7}y^{\frac{7}{3}}]_0^8$$
$$V = 2\pi\{[16(8) - 6(8)^{\frac{4}{3}} - (8)^2 + \tfrac{3}{7}(8)^{\frac{7}{3}}] - [16(0) - 6(0)^{\frac{4}{3}} - (0)^2 + \tfrac{3}{7}(0)^{\frac{7}{3}}]\}$$
$$V = 2\pi(\tfrac{160}{7})$$
$$V = \tfrac{320}{7}\pi cu.units$$

Plane Areas in Polar Coordinates

Find the area of the region bounded by the graph of r = 2 + 2 cos θ.

Solution.

By assigning various values of θ and finding the corresponding values of r, we obtain the graph below,



The curve is symmetric with respect to the polar axis, we take the limits from 0 to π. The area of the entire region is multiplied by 2: Thus the required area is

$$A = 2\int_0^{\pi}\frac{1}{2}(2+2\cos\theta)^2\,d\theta$$

$$= 4\int_0^{\pi}\left(1+2\cos\theta+\cos^2\theta\right)d\theta$$

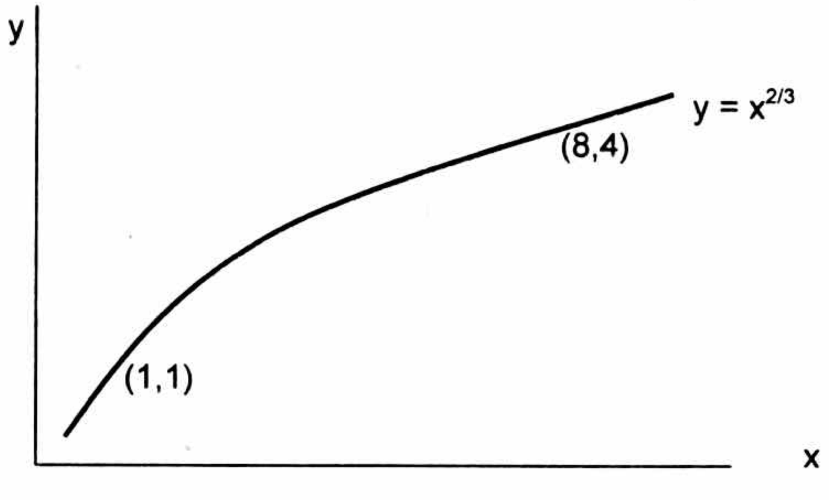
$$= 4\left[\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi}$$

$$= 4\left(\pi+0+\frac{1}{2}\pi+0-0\right)$$

$$= 6\pi$$

Length of a Plane Curve (s)

1. Find the length of the arc of the curve $y = x^{2/3}$ from the point (1,1) to (8,4)



Solution. Given $f(x) = x^{2/3}, f'(x) = \frac{2}{3}x^{-1/3}$, using equation (1),

$$s = \int_1^8 \sqrt{1+\left(\frac{2}{3x^{-1/3}}\right)^2}\,dx$$

$$= \int_1^8 \sqrt{1+\left(\frac{4}{9x^{2/3}}\right)}\,dx$$

$$= \frac{1}{3}\int_1^8 \frac{\sqrt{9x^{2/3}+4}}{x^{1/3}}\,dx$$

To evaluate, let $u = 9x^{2/3} + 4$, then du = $6x^{-1/3}dx$
When x = 1, u = 13, when x = 8, u = 40

$$\begin{aligned}
s &= \frac{1}{18}\int_{3}^{40} u^{1/2}\,du \\
&= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{13}^{40} \\
&= \frac{1}{27}\left(40^{2/3} - 13^{2/3}\right) \\
&\approx 7.6
\end{aligned}$$

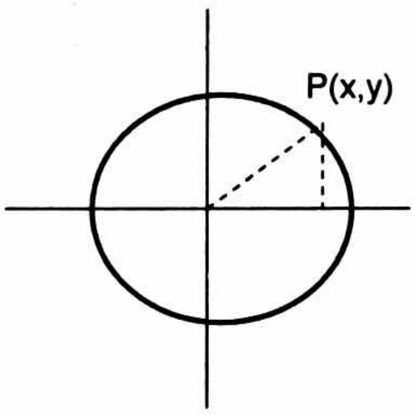
or using equation (2)

Given $y = x^{2/3}$, solving for x, x>0, $x = y^{3/2}$. Let $g(y) = y^{3/2}, g'(y) = \frac{3}{2}y^{1/2}$, then

$$\begin{aligned}
s &= \int^{4} \sqrt{1 + \left(\frac{3}{2}y^{1/2}\right)^2}\,dy \\
&= \int^{4} \sqrt{1 + \left(\frac{9}{4}y\right)}dy \\
&= \frac{1}{2}\int^{4} \sqrt{4 + 9y}\,dy \\
&= \frac{1}{18}\left[\frac{2}{3}(4 + 9y)^{3/2}\right]_{1}^{4} \\
&= \frac{1}{27}\left(40^{3/2} - 13^{3/2}\right) \\
&\approx 7.6
\end{aligned}$$

2.The coordinate (x,y) of a point on a circle of radius r can be expressed in terms of the central angle θ as

$$x = r\cos\theta \qquad\qquad y = r\sin\theta$$



Solution.

The point P(x,y) moves around the circle as θ varies from 0 to 2π, so the circumference of the circle is

$$C = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Since,

$$\frac{dx}{d\theta} = -r\sin\theta, \frac{dy}{d\theta} = r\cos\theta$$

So that

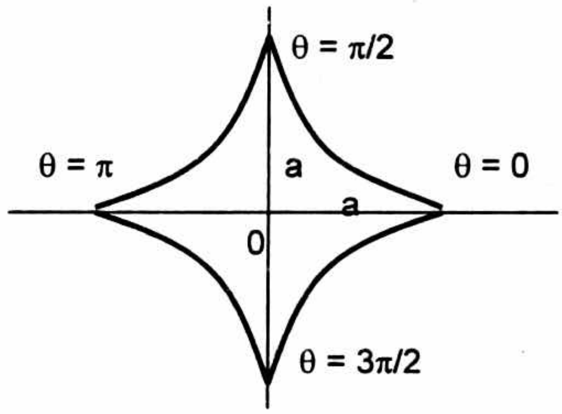
$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = r^2\left(\sin^2\theta + \cos^2\theta\right) = r^2$$

Hence,

$$C = \int_0^{2\pi} r\,d\theta = r[\theta]_0^{2\pi} = 2\pi r$$

3. The coordinate of the point $P(x, y)$ on the four-cusped hypocycloid are given by

$$x = a\cos^3\theta \qquad\qquad y = a\sin^3\theta$$



The hypocloid $x = a\cos^3\theta, y = a\sin^3\theta,\ 0 \le \theta \le 2.$

Find the length of the curve.

Solution.

θ varies from 0 to π/2, P traces out the portion of the curve in the first quadrant. Thus, the total length of the curve,

$$s = 4\int_{\theta=0}^{\theta=\pi/2} \sqrt{dx^2 + dy^2}$$

From the equation of the curve,

$$\frac{dx}{d\theta} = -3a\cos^2 \sin\theta, dx = 3a\cos\theta\sin\theta(-\cos\theta)d\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2 \cos\theta, dy = 3a\cos\theta\sin\theta(\sin\theta)d\theta$$

Hence,

$$ds^2 = dx^2 + dy^2 = (3a\cos\theta\sin\theta)^2(\cos^2\theta + \sin^2\theta)(d\theta)^2$$

or

$$ds^2 = (3a\cos\theta\sin\theta d\theta)^2$$

Hence for θbetween 0 to π/2,

$$ds = 3a\cos\theta\sin\theta d\theta$$

and

$$s = 4\int_0^{\pi/2} 3a\cos\theta\sin\theta d\theta$$

To evaluate this integral, let

$$u = \sin\theta, du = \cos\theta d\theta$$

and,

$$\int\cos\theta\sin\theta d\theta = \int u du = \frac{1}{2}u^2 = \frac{1}{2}\sin^2\theta$$

so that

$$s = (12a)\left(\frac{1}{2}\sin^2\theta\right)_0^{\pi/2} = 6a$$

Area of Surface of Revolution (S)

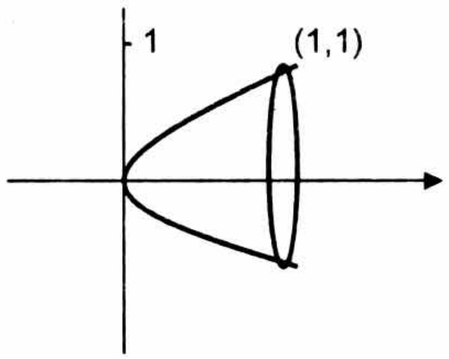
The **surface of revolution** is that surface obtained by revolving an arc about a given axis of revolution.

The formula is given by

$$S = 2\pi\int r ds$$

Examples

1. Find the area of the surface generated by revolving about the x-axis the arc $C = \{(x, y) | 0 \le y \le 1, x = y^2\}$



Solution.

Since $x = y^2, \dfrac{dx}{dy} = 2y$ and $ds = \sqrt{1 + \left(\dfrac{dx}{dy}\right)^2}\, dy = \sqrt{1 + (2y)^2}\, dy$

Thus,

$$S = 2\pi \int_0^1 y\sqrt{1 + 4y^2}\, dy$$

To evaluate this integral, let

u = 1 + 4y^2 ;   du = 8ydy or $\dfrac{1}{8} du = ydy$

So

$$\begin{aligned} S &= 2\pi \int_c^d \sqrt{u} \cdot \frac{1}{8} du \\ &= \frac{1}{4}\pi \int_c^d u^{1/2} du \\ &= \frac{\pi}{4}\left(\frac{2}{3} u^{3/2}\right)_c^d \\ &= \frac{\pi}{6}\left(1 + 4y^2\right)^{3/2} \Big|_0^1 \\ &\approx 5.33 sq.units \end{aligned}$$

2. The circle

$$x^2 + y^2 = r^2$$

is revolved about the x – axis. Find the area of the sphere generated.

Solution. We write

$$ds = 2\pi y ds$$
$$ds = \sqrt{dx^2 + dy^2}$$

and use

$$x = r\cos\theta \qquad y = r\sin\theta$$

to represent the circle. Then,

$$dx = -r\sin\theta d\theta, dy = r\cos\theta d\theta$$

so that

$$ds = rd\theta$$

and

$$dS = 2\pi y ds = 2\pi(r\sin\theta)rd\theta$$
$$= 2\pi r^2 \sin\theta d\theta$$

Hence,

$$S = \int 2\pi r^2 \sin\theta d\theta$$
$$= 2\pi r^2[-\cos\theta]_0^{\pi}$$
$$= 4\pi r^2$$

Physical Applications of Integration

The Mean Value of a Function

1. Find the mean value of x^2 between x = 1 and x = 3.

Solution.

$$= \frac{1}{3-1}\int_1^3 x^2 dx$$

The mean value ŷ $= \frac{1}{2}\left[\frac{x^3}{3}\right]_1^3$

$$= \frac{27-1}{6}$$
$$= \frac{13}{3}$$

2. Find the average velocity for a free falling body starting from rest,

$$s=\frac{1}{2}gt^2, \qquad v=gt, \qquad v=\sqrt{2gs}$$

a) with respect to t and b) with respect to s, from $t_1 = 0$, $s_1 = 0$, to $t_2 > 0$, $s_2 = ½\, g{t_2}^2$.

Solution.

a. $$\left(v_{av}\right)_t=\frac{1}{t_2-0}\int_0^{t_2} gt\,dt=\frac{1}{2}gt_2=\frac{1}{2}v_2$$

b. $$\left(v_{av}\right)_s=\frac{1}{s_2-0}\int_0^{s_2}\sqrt{2gs}\,ds=\frac{2}{3}\sqrt{2gs_2}=\frac{2}{3}v_2$$

Mean value of a periodic function

3. Find the mean value of $\cos^2 pt$

Solution

The period of the function cos pt is $\dfrac{2\pi}{p}$

$$\text{Mean value}=\frac{1}{\frac{2\pi}{p}-0}\int_0^{2\pi/p}\cos^2 pt\,dt$$

$$=\frac{p}{2\pi}\int_0^{2\pi/p}\frac{(1+\cos 2pt)}{2}dt$$

$$=\frac{p}{4\pi}\left[t+\frac{\sin 2pt}{2p}\right]_0^{2\pi/p}$$

$$=\frac{p}{4\pi}\left[\frac{2\pi}{p}+\frac{1}{2p}\sin 4\pi\right]$$

$$=\frac{p}{4\pi}\times\frac{2\pi}{p}$$

$$=\frac{1}{2}$$

The Mean Square Value of a Function

$$= \frac{1}{b-a}\int_a^b y^2 dx$$

The Root Mean Square (r.m.s.) value of a function

$$\text{r.m.s value} = \sqrt{\frac{1}{b-a}\int_a^b y^2 dx}$$

1. Find the r.m.s. value of $i = I\sin pt$ over the range t = 0 and t = $\frac{2\pi}{p}$.

Solution.

$$\text{Mean square value} = \frac{1}{\frac{2\pi}{p}}\int_{0a}^{2\pi/p}(I\sin pt)^2 dt$$

$$= \frac{p}{2\pi}\int I^2 \sin^2 ptdt$$

$$= \frac{pI^2}{2\pi}\int_0^{2\pi/p}\frac{1-\cos 2pt}{2}dt$$

$$= \frac{pI^2}{2\pi}\left[t - \frac{\sin 2pt}{2p}\right]_0^{2\pi/p}$$

$$= \frac{pI^2}{2\pi}\left[\frac{2\pi}{p} - 0\right]$$

$$= \frac{I^2}{2}$$

Hence r.m.s. value = $\sqrt{\frac{I^2}{2}} = \frac{I}{\sqrt{2}} = 0.707I$

2. If $V = V_0 \sin \omega t$ and $C = C_0 \sin(\omega t - \alpha)$, find the mean value of the power product VC

Solution.

$$VC = V_0 \sin \varpi t C_0 \sin(\omega t - \alpha)$$

$$= \frac{V_0 C_0}{2}\left[\cos\alpha - \cos(2\omega t - \alpha)\right]$$

The period ov VC is $\frac{2\pi}{\omega}$

Hence mean value = $\frac{1}{\frac{2\pi}{\omega}} \int \frac{V_0 C_0}{2}\left[\cos\alpha - \cos(2\omega t - \alpha)\right] dt$

$$= \frac{\omega V_0 C_0}{4\pi}\left[t\cos\alpha - \frac{\sin(2\omega t - \alpha)}{2\omega}\right]_0^{2\pi/\omega}$$

$$= \frac{\omega V_0 C_0}{4\pi}\left[\frac{2\pi}{\omega}\cos\alpha - \frac{\sin(4\pi - \alpha) - \sin(-\alpha)}{2\omega}\right]$$

$$= \frac{\omega V_0 C_0}{4\pi}\left(\frac{2\pi}{\omega}\cos\alpha - 0\right)$$

$$= \frac{1}{2} V_0 C_0 \cos\alpha.$$

Work Done by a Variable Force

1. A body moves from rest under the action of a direct force given by $P = \frac{16}{x+2} N$, where x is the distance in meters from the starting point. Find the total work done in moving a distance 4 m, and find the mean value of the force over this distance.

Solution.

Work done = $\int_0^4 P dx = \int_0^4 \frac{16}{x+2} dx$

$$= 16\left[\ln(x+2)\right]_0^4$$

$$= 16\left[\ln 6 - \ln 2\right]$$

$$= 16 \ln\frac{6}{2}$$

= 16 ln 3 = 17.58J

Mean value force = $\frac{17.58 Nm}{4m} = 4.39N$

2. A buffer spring is compressed a distance 0.4m. Find the additional work which must be done to compress it a further 0.2m if the spring obeys Hooke's law and its stiffness is such that a force of 3kN will compress it 1 cm.

Solution.

Since the spring obeys Hooke's law

Force P $\propto$ compression x, that is,

$P = Sx$

Work done = $\int_{0.4}^{0.6} P dx = \int_{0.4}^{0.6} Sx dx$

$$= S\left[\frac{x^2}{2}\right]_{0.4}^{0.6} = \frac{S}{2}[0.36 - 0.16]$$

$$= 0.10 Sm^2$$

When x = 0.01m , P = 3kN

3kN = S(0.01m)

S = 300kN m^{-1}

Work done = 0.10(300kNm^{-1})m^2

= 30kNm

= 30. kJ

3. A gas expands according to the law $pv^{1.2} = C$. Initially the pressure is 250kNm^{-2} when the volume is 1 m^3. Find the work done by the gas in expanding to twice its original volume.

Solution.

$pv^{1.2} = C = 250 \times 1^{1.2}$ (kNm)

C = 250 and p = $\frac{250}{v^{1.2}}$

Work done = $\int_1^2 p dv = \int_1^2 \frac{250}{v^{1.2}} dv$

$$= \frac{250}{-0.2}\left[\frac{1}{v^{0.2}}\right]_1^2$$

$$= 1{,}250\left[1 - \frac{1}{2^{0.2}}\right]$$

$$= 1{,}250[1 - 0.87055]$$

$$= 161.81 kJ$$

4. A trapezoid is submerged vertically in water with its upper edge 3 ft below the surface and its lower edge 9 ft below the surface. If the upper and lower edges are 5 ft and 7 ft, find the total force on one surface of the trapezoid.

Solution.

By similar triangle,

$$\frac{l-5}{2}=\frac{h-3}{6}$$

so that

$$l=\frac{h+12}{3}$$

Then the force is

$$F=\int_3^9 wh\left(\frac{h+12}{3}\right)dh$$

$$=\frac{1}{3}\int_3^9 \left(wh^2+12wh\right)dh$$

$$=\frac{1}{3}\left[\frac{wh^3}{3}+wh^2\right]_3^9$$

$$=102w$$

Since,

$$w=62.5lb/ft^3=\frac{1}{32}tons/ft^3$$

we have

$$F=(102)\left(\frac{1}{32}\right)=3.1875tons$$

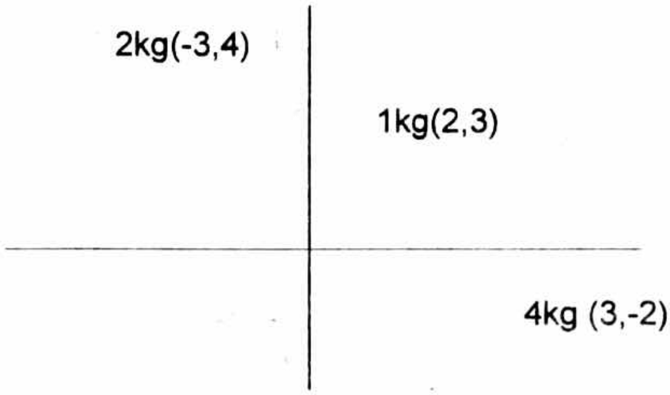
Mass and Center of Gravity

1. Find the moment of m located at (2,4)

Solution.

M_x = mass or moment of m with respect to x
= M_d = my
= m(4)
= 4m
M_y = mass or moment of m with respect to y
= M_d = mx
= m(2)
= 2m

1. Find the centroid of the following systems 1, 2, 4 kilograms located at points (2,3), (-3,4) and (3,-2), respectively.



Solution.

The total mass is

$$\begin{aligned} M &= \sum m_i \\ &= 1+2+4 \\ &= 7kg \end{aligned}$$

The mass with respect to x - axis is

$$\begin{aligned} M_x &= \sum m_i y_i \\ &= 1(3)+2(4)+4(-2) \\ &= 3 \end{aligned}$$

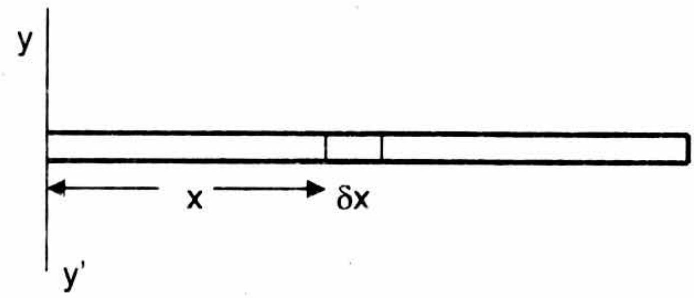
The mass with respect to the y-axis is

$$\begin{aligned} M_y &= \sum m_i x_i \\ &= 1(2)+2(-3)+4(3) \\ &= 8 \end{aligned}$$

The center of mass is $\left(\frac{8}{7}, \frac{3}{7}\right)$

$$\begin{aligned} \bar{x} &= \frac{M_y}{M} = \frac{8}{7} \\ \bar{y} &= \frac{M_x}{M} = \frac{3}{7} \end{aligned}$$

3.The mass per unit length of a rod at a ditance x m from one end is $\left(4+\frac{x^2}{4}\right)kgm^{-1}$ .If the rod is 6 m long find its total mass.



Solution.

value $\left(4+\frac{x^2}{4}\right)kgm^{-1}$

Mass of element = $\left(4+\frac{x^2}{4}\right)\delta x$

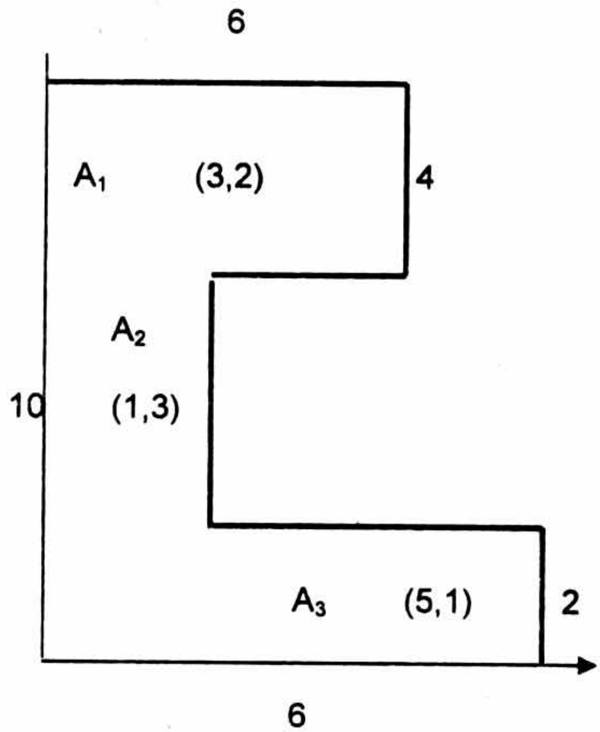
Total mass of the rod is the sum of the masses of all such elements making up the rod.

Total mass = $\sum_{x=0}^{x=6}\left(4+\frac{x^2}{4}\right)\delta x$

$$= \int_0^6 \left(4+\frac{x^2}{4}\right)dx$$
$$= \left[4x+\frac{x^3}{12}\right]_0^6$$
$$= 42kg$$

Center of Gravity

1. Find the centroid of the metal plate



Solution.

$A_1 = 4 \times 6 = 24$

$A_2 = 4 \times 2 = 8$

$A_3 = 2 \times 6 = 12$

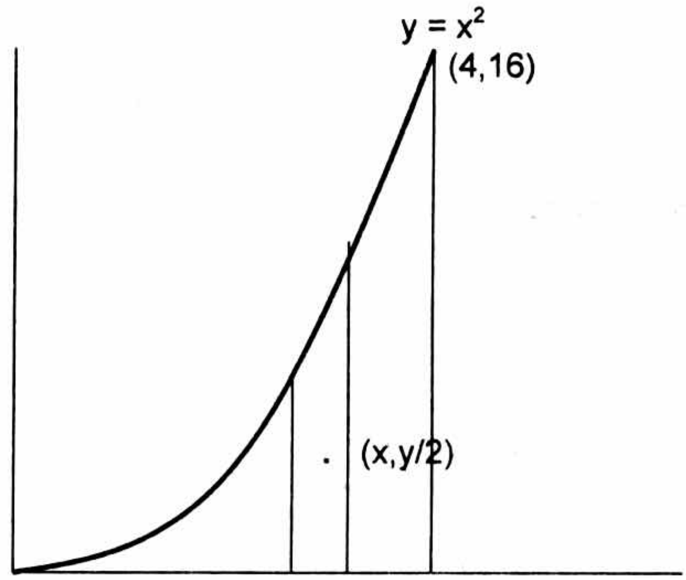
$A = 24 + 8 + 12 = 44$

$$\bar{x} = \frac{\sum ax}{A}$$
$$= \frac{24(3) + 8(1) + 12(5)}{44}$$
$$= \frac{140}{44}$$
$$= \frac{35}{11}$$
$$\bar{y} = \frac{\sum ay}{A}$$
$$= \frac{24(2) + 8(3) + 12(1)}{44}$$
$$= \frac{84}{44}$$
$$= \frac{21}{11}$$

The centroid is $\left(\frac{35}{11}, \frac{21}{11}\right)$

1. Find the centroid of the area bounded by the curve $y = x^2$ and the ordinate $x = 0$ and $x = 4$.

Solution.



$$\bar{x} = \frac{\int_a^b xy\,dx}{\int_a^b y\,dx} = \frac{\int_0^4 x^3\,dx}{\int_0^4 x^2\,dx}$$

$$= \frac{\frac{1}{4}\left[x^4\right]_0^4}{\frac{1}{3}\left[x^3\right]_0^4} = \frac{3(4^4)}{4(4^3)}$$

$$= 3$$

$$\bar{y} = \frac{\int_0^4 \frac{y^2}{2}\, dx}{\int_0^4 y dx} = \frac{\frac{1}{2}\int_0^4 x^4 dx}{\frac{64}{3}}$$

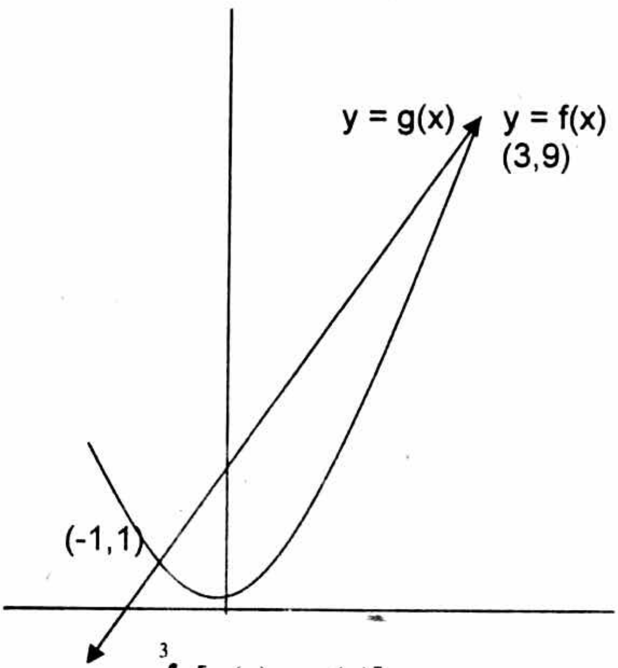
$$= \frac{\frac{1}{10}\left[x^5\right]_0^4}{\frac{64}{3}} = \frac{\frac{1,024}{10}}{\frac{64}{3}}$$

$$= \frac{24}{5}$$

The centroid is at the point $(3, \frac{24}{5})$.

2. Find the centroid of the region bounded by the curve $y = x^2$ and $y = 2x + 3$.

Solution.



$$\bar{x} = \frac{\int_{-1}^{3} x[g(x) - f(x)]dx}{\int_{-1}^{3} [g(x) - f(x)]dx}$$

$$= \frac{\int_{-1}^{3} x\left[2x+3-x^2\right]dx}{\int_{-1}^{3}\left[2x+3-x^2\right]dx}$$

$$= \frac{\left[2\frac{x^3}{3}+3\frac{x^2}{2}-\frac{x^4}{4}\right]_{-1}^{3}}{\left[x^2+3x-\frac{x^3}{3}\right]_{-1}^{3}}$$

$$= \frac{\frac{32}{3}}{\frac{32}{3}} = 1$$

$$\bar{x} = \frac{\frac{1}{2}\int_{-1}^{3}\left[g(x)+f(x)\right]\left[g(x)-f(x)\right]dx}{\int_{-1}^{3}\left[g(x)-f(x)\right]dx}$$

$$= \frac{\frac{1}{2}\int_{-1}^{3}\left[2x+3+x^2\right]\left[2x+3-x^2\right]dx}{\frac{32}{3}}$$

$$= \frac{\frac{1}{2}\int_{-1}^{3}\left[4x^2+12x+9-x^4\right]dx}{\frac{32}{3}}$$

$$= \frac{\frac{1}{2}\left[4\frac{x^3}{3}+6x^2+9x-\frac{x^5}{5}\right]_{-1}^{3}}{\frac{32}{3}}$$

$$= \frac{\frac{544}{15}}{\frac{32}{3}}$$

$$= \frac{17}{5}$$

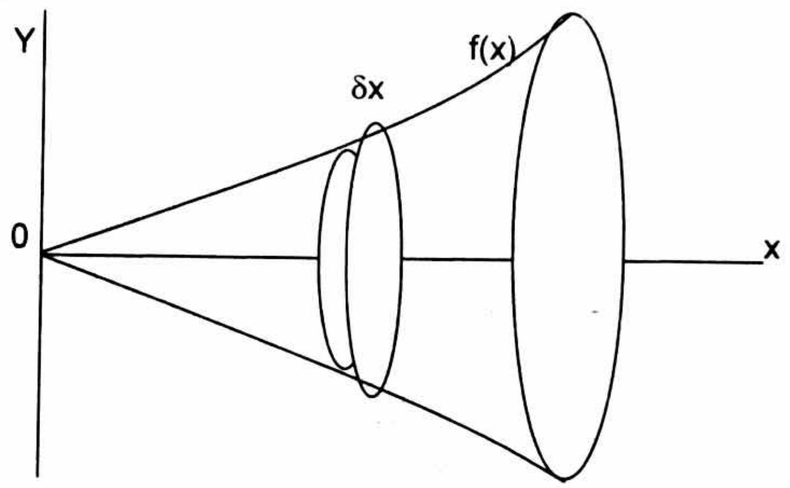
Hence the centroid is at the point $\left(1, \frac{17}{5}\right)$

Center of gravity of a Solid Revolution

1. Find the center of gravity of the solid of revolution generated when the curve $y = x^2$, from ordinates $x = 0$ to $x = 3$, revolve about the x-axis.

Solution.

Let the volume be divided into a large number of discs by planes perpendicular to the x – axis, distance δx apart.



Consider a sample element of radius y, thickness δx as shown in the figure above. If w is the weight per unit volume

Weight of disc = $\pi y^2\ \delta x \times w$

By taking moments about the y- axis of the weights of all the discs, then by the principle of moments

(total weight) x $\bar{x} = \sum_{x=0}^{x=3} \pi y^2 \delta x \times wx$

Since total weight = volume x w = $w \int_0^3 \pi y^2 dx$

$$\bar{x} = \frac{\int_0^3 \pi x y^2 dx}{\int_0^3 \pi y^2 dx}$$

Thus $$= \frac{\int_0^3 x^5 dx}{\int_0^3 x^4 dx}$$

$$= \frac{\frac{1}{6}\left[x^6\right]_0^3}{\frac{1}{5}\left[x^5\right]_0^3}$$

$$= \frac{5}{2}$$

The center of gravity of solid of revolution is at point $\left(\frac{5}{2},0\right)$.

Theorem of Pappus

1. Find the volume and surface area of the torus (doughnut) generated by rotating a circle of radius r about an axis in its plane at a distance a from its center.

Solution. The center of the circle is the center of gravity, and this travels a distance $2\pi a$. The area of the circle is $2\pi r^2$, so the volume of the torus is

V = $(2\pi a)(\pi r^2) = 2\pi^2 a r^2$

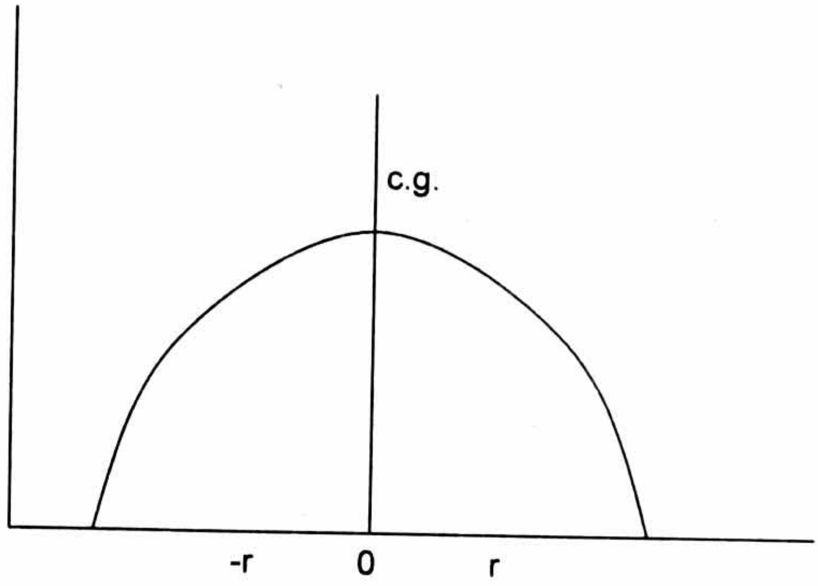
The area of surface is

S = $(2\pi a)(2\pi r) = 4\pi^2 a r$

2. Use the first Theorem of Pappus and the fact that the volume of a sphere of radiusr is

$$V = \frac{4}{3}\pi r^3$$

Solution. Note that we generate a sphere by rotating a semi-circle about its diameter. Then the axis of revolution does not intersect the area



and since

$$A = \pi r^2$$

while

$$V = 2\pi \bar{y} A$$

we find

$$\bar{y} = \frac{V}{2\pi A} = \frac{\frac{4}{3}\pi r^3}{2\pi \cdot \frac{1}{2}\pi r^2} = \frac{4}{3\pi} r.$$

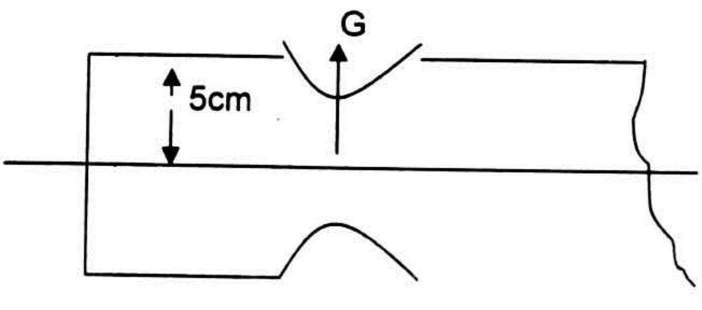
3. A steel shaft 10 cm diameter has a semicircular groove turned out of it to a depth of 2 cm. Calculate the volume of metal removed.

Solution.

Area of the semi-circle = $\frac{1}{2}\pi 2^2 = 2\pi cm^2$

Distance of the centroid of semicircular area from its center = $\frac{4r}{3\pi} = \frac{4 \times 2}{3\pi} = \frac{8}{3\pi} cm$

Distance of centroid from axis = $\bar{y} = \left(5 - \frac{8}{3\pi}\right) cm$



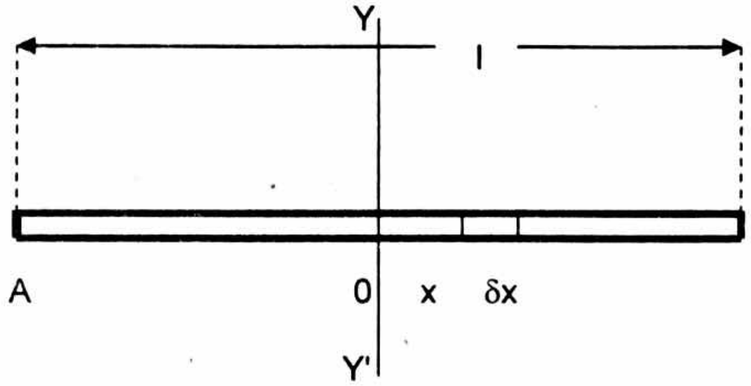
Volume of metal remove = area x $2\pi \bar{y}$

$$= (2\pi)\left(2\pi\left(5 - \frac{8}{3\pi}\right)\right)$$

$$= 163.88 cm^3$$

Moments of Inertia

1. About an axis through its center normal to its length.
   Let m be the mass per unit length of the rod of length l.
   Consider a small element of length δx at a distance x from axis YY'. See figure.



Mass of element = mδx

M.I. of element about YY' = $(m\delta x)x^2 = mx^2\delta x$

M.I. of rod $I_{YY'}$ = $\sum_{x=-\frac{1}{2}l}^{x=+\frac{1}{2}l} mx^2 \delta x$

$$= \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} mx^2 dx = m\left[\frac{x^3}{3}\right]_{-\frac{1}{2}l}^{+\frac{1}{2}l}$$

Since the total mass of the rod M = ml

Then $I_{YY'} = \frac{Ml^2}{12}$ and $k^2_{YY'} = \frac{l^2}{12}$

2. About a perpendicular axis at one end.

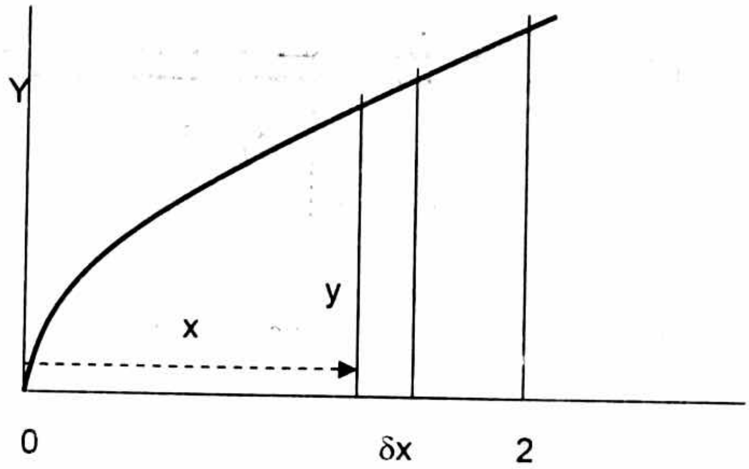
In this case the distance x is measured from the axis AA' and the limits of integration become *0 to l*.

$$I_{AA'} = \int_0^l mx^2 dx = \frac{ml^3}{3}$$

$$= \frac{Ml^2}{3} \text{ and } k^2{}_{AA'} = \frac{l^2}{3}$$

Second Moment of Area of a Rectangle

Find the radius of gyration about boyh Ox and Oy of the area in the first quadrant, between the curve $y^2 = 4x$, the x-axis and the ordinate $x = 2$.



1. about the y-axis

$$k^2Oy = \frac{\int_a^b x^2 y dx}{\int_a^b y dx} = \frac{\int_0^2 2x^{\frac{5}{2}} dx}{\int_0^2 2x^{\frac{1}{2}} dx}$$

$$= \frac{\frac{4}{7}\left[x^{\frac{7}{2}}\right]_0^2}{\frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^2} = 1.71$$

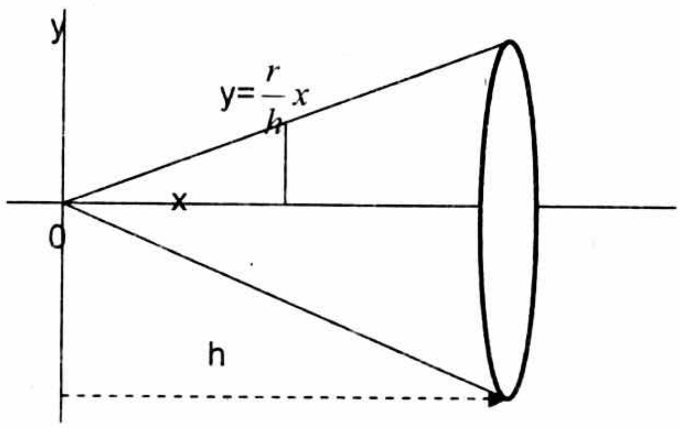
2. about the x axis

$$k^2_{Ox} = \frac{\int_a^b \frac{1}{3} y^3 dx}{\int_a^b y dx} = \frac{\int_0^2 \frac{8}{3} x^{\frac{3}{2}} dx}{\int_0^2 2x^{\frac{1}{2}} dx}$$

$$= \frac{\frac{16}{15}\left[x^{\frac{5}{2}}\right]_0^2}{\frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^2} = 1.60$$

The Moment of Inertia of a Solid of Revolution

Find the moment of inertia of a right circular cone about its axis.



Let the cone be placed as in the figure, with its axis along the x-axis.
Treating it as a solid of revolution the equation of the line generating it is given by $\frac{y}{x} = \frac{r}{h}$. That is y = $\frac{r}{h}x$.

Mass $M = \frac{1}{3}\pi r^2 h\rho$

The moment of inertia

$$= \int_0^h \frac{\rho\pi}{2} y^4 dx$$

$$= \frac{\rho\pi}{2} \int \frac{r^4}{h^4} x^4 dx$$

$$= \frac{\rho\pi r^4}{2h^4} \left[\frac{h^5}{5} \right]$$

$$= \frac{\rho\pi r^4 h}{10}$$

$$= \frac{\rho\pi r^2 h}{3} \times \frac{3}{10} r^2$$

$$= M \frac{3}{10} r^2$$

Double Integrals

1. Find the volume of the solid whose base is in the xy-plane and is the triangle bounded by the x-axis, the line y = x, and the line x = 1, while the top of the solidis in the plane

   Z = F9x,y) = 3 – x – y

   Solution. The volume dV

   dV = (3 – x – y) dydx

   For any value x between 0 and 1, y may vary from y = 0 to y = x. Hence

$$V = \int_0^1 \int_0^x (3 - x - y) dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx$$

$$= \frac{3x^2}{2} - \frac{x^3 2}{2} \Bigg]_{x=0}^{x=1}$$

$$= 1$$

2. Evaluate the double integral

$$\int_1^3 \int_{-1}^2 (3y - 2x^2) dx dy$$

Solution.

$$\int_1^3\int_{-1}^2 (3y-2x^2)dxdy = \int_1^3\left[\int_{-1}^2 (3y-2x^2)dx\right]dy$$

$$= \int_1^3\left[3xy-\frac{2x^3}{3}\right]_{x=-1}^{x=2} dy$$

$$= \int_1^3 (9y-6)dy$$

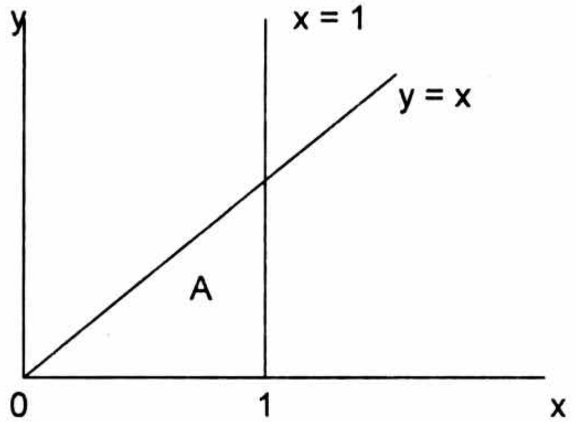
$$= \frac{9}{2}y^2-6y\Bigg]_{y=1}^{y=3}$$

$$= 24$$

3. Calculate

$$\iint_A \frac{\sin x}{x}\,dA,$$

where A is the triangle in the xy-plane bounded by the x – axis, the line y = x and the line x = 1.



Solution.

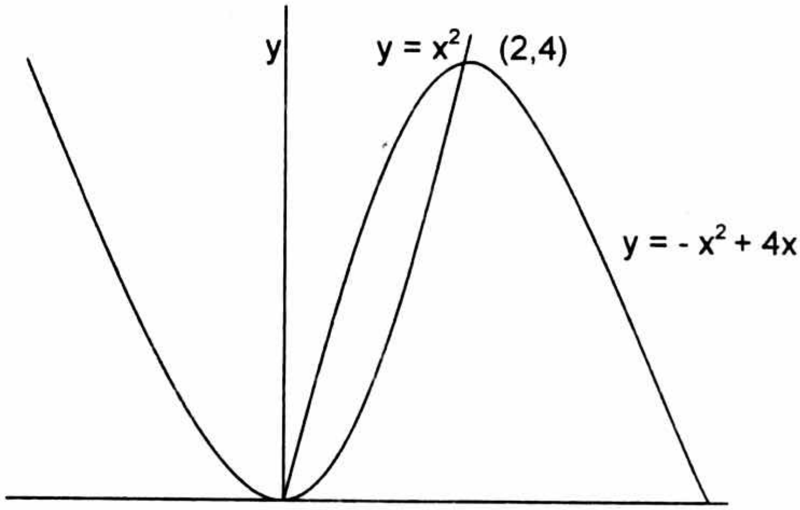
$$\int_0^1\left(\int_0^x \frac{\sin x}{x}dy\right)dx = \int_0^1\left[y\frac{\sin x}{x}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \sin x dx$$

$$= -\cos(1)+1 \approx 0.46$$

Area by Double Integration

1. Find tby double integration the area of the region in the xy-plane bounded by the curve $y = x^2$ and $y = 4x - x^2$.



Solution.

$$A = \iint_R dydx$$

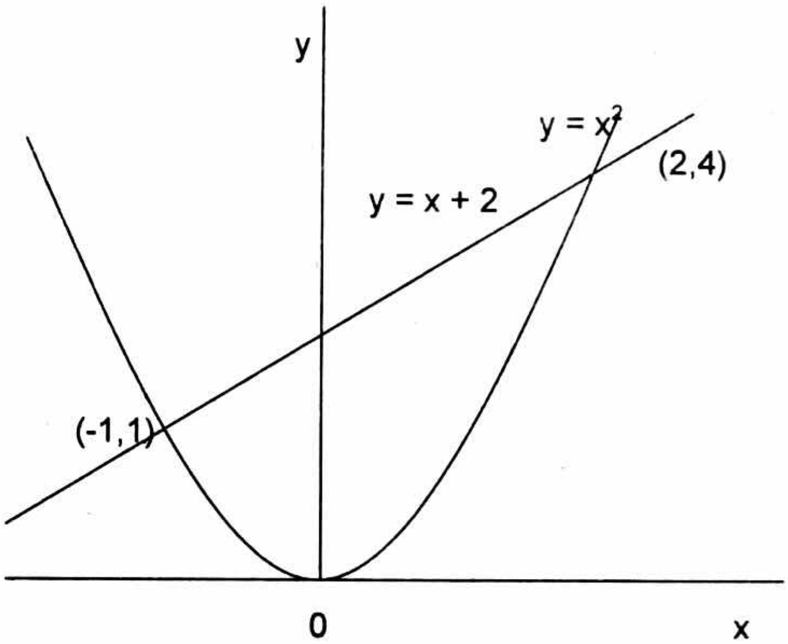
$$= \int_0^2 \int_{x^2}^{4x=x} dydx$$

$$= \int_0^2 \left(4x - x^2 - x^2\right)dx$$

$$= 2x^2 - \frac{2}{3}x^3 \Big]_0^2$$

$$= \frac{8}{3}$$

2. Find the area bounded by the parabola $y = x^2$ and the line $y = x + 2$



Solution.

$$A = \int_{-1}^{2}\int_{x^2}^{x+2} dy dx$$

$$= \int_{-1}^{2} y\Big]_{y=x^2}^{y=x+2} dx$$

$$= \int_{0}^{2}\left(x + 2 - x^2\right)dx$$

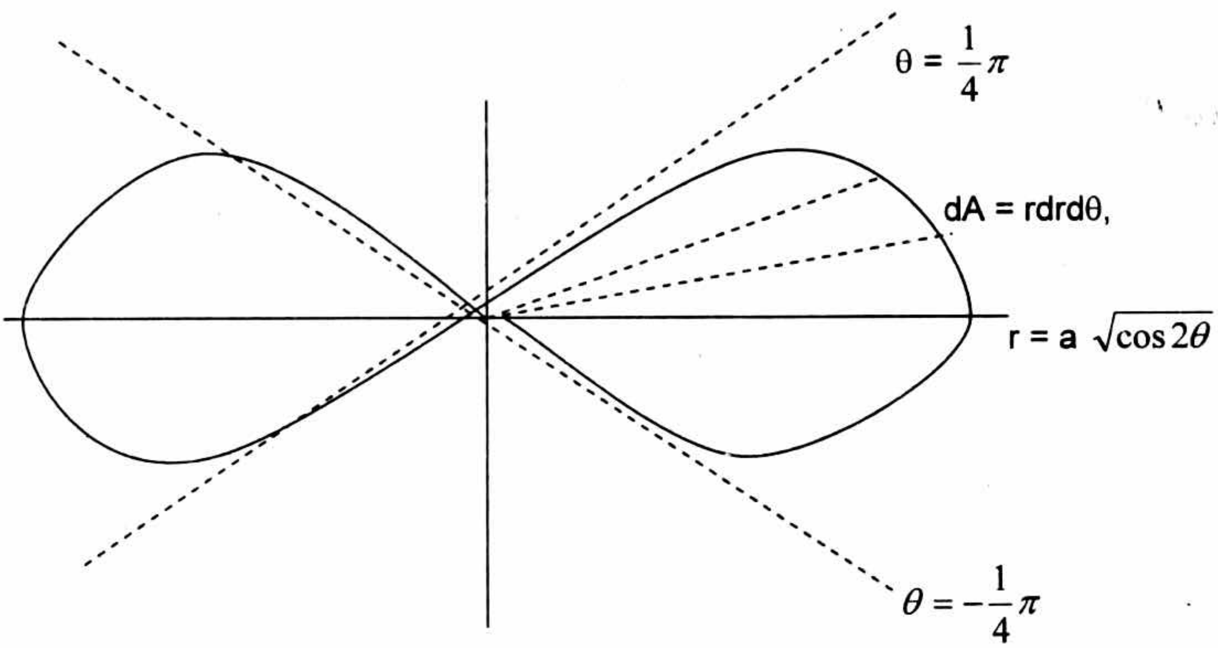
$$= \frac{x^2}{2} + 2x - \frac{x^3}{3}\Bigg]_0^2$$

$$= \frac{9}{2}$$

The Double Integral In Polar Coordinates

Find the area of the region in the xy plane bounded by the lemniscate $r^2 = a^2 \cos 2\theta$,

Solution.



The required area is

$$4\int_{\theta=0}^{\frac{\pi}{4}}\int_{r=0}^{a\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_{r=0}^{a\sqrt{\cos 2\theta}} d\theta$$

$$= 2\int_{\theta=0}^{\frac{\pi}{4}} a^2 \cos 2\theta\, d\theta$$

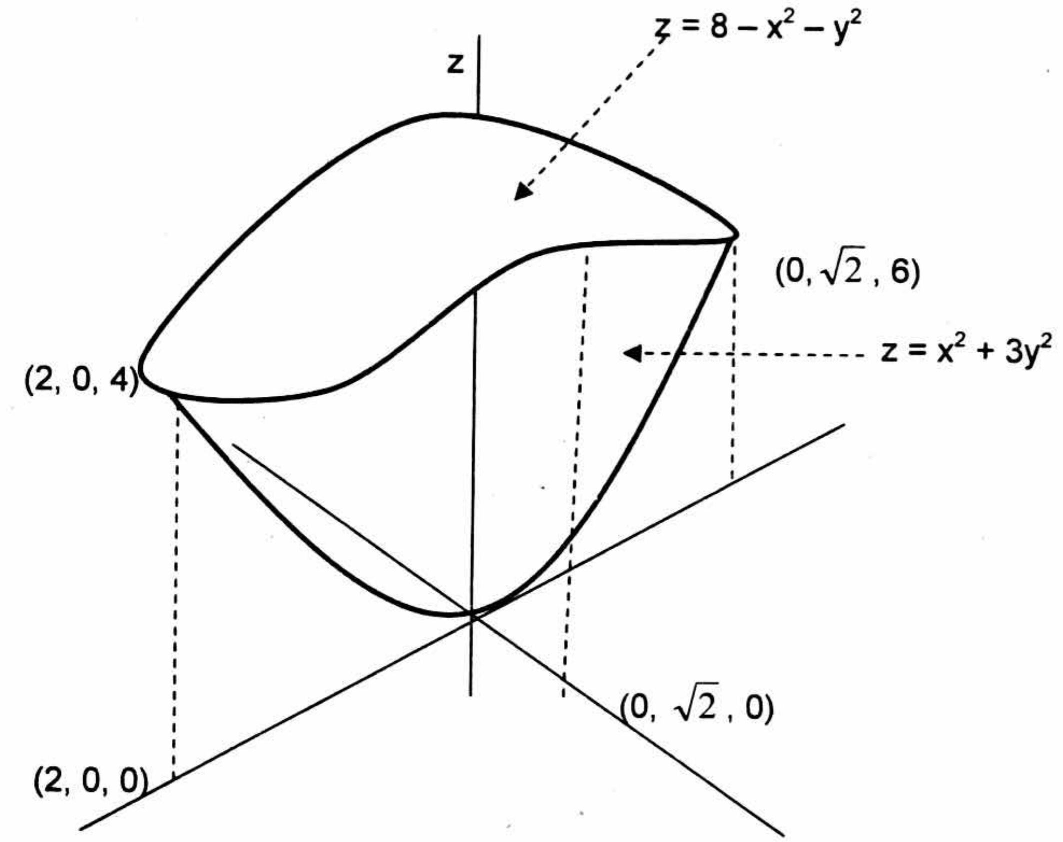
$$= \left[a^2 \sin 2\theta\right]_{\theta=0.}^{\frac{\pi}{4}}$$

$$= a^2$$

Triple Integrals. Volume

1. Find the volume enclosed between the two surfaces

$$Z = 8 - x^2 - y^2 \text{ and } z = x^2 + 3y^2.$$



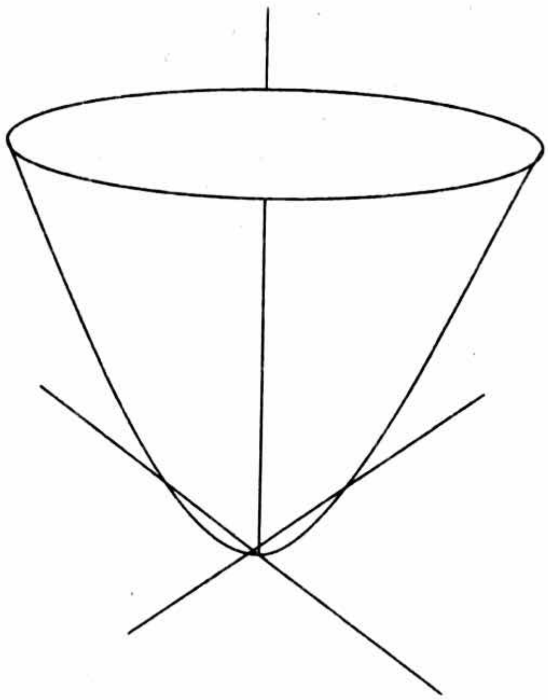
Solution. The two surfaces in the figure intersect on the elliptic cylinder

$$x^2 + 2y^2 = 4$$

If we integrate first with respect to y, holding x and dx fixed, y varies from $-\dfrac{\sqrt{4-x^2}}{2}$ to $\dfrac{\sqrt{4-x^2}}{2}$. The x varies from – 2 to 2. Thus

$$\begin{aligned}
V &= \int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}}\int_{x^2+3y^2}^{8-x^2-y^2} dzdydx \\
&= \int_{-2}^{2}\int_{-\frac{\sqrt{4-x^2}}{2}}^{\frac{\sqrt{4-x^2}}{2}}\left(8-2x^2-4y^2\right)dydx \\
&= \int_{-2}^{2}\left[2\left(8-2x^2\right)\sqrt{\frac{4-x^2}{2}}-\frac{8}{3}\left(\frac{4-x^2}{2}\right)^{\frac{3}{2}}\right]dx \\
&= \frac{4\sqrt{2}}{3}\int_{-2}^{2}\left(4-x^2\right)^{\frac{3}{2}}dx \\
&= 8\pi\sqrt{2}
\end{aligned}$$

2. Find the solid above the xy-plane bounded by the elliptic paraboloid $z = x^2 + 4y^2$ and the cylinder $x^2 + 4y^2 = 4$.



Solution.

The z limit are 0 to $x^2 + 4y^2$. The y limits for ¼ of the volume are from 0 to $\frac{1}{2}\sqrt{4-x^2}$. The x limits for the first octant are from 0 to 2. We evaluate the triple integral byan iterated integral and obtain

$$V = \int_0^2 \int_0^{\frac{1}{2}\sqrt{4-x^2}} \int_0^{x^2+4y^2} dzdydx$$

$$= 4\int_0^2 \int_0^{\frac{1}{2}\sqrt{4-x^2}} \left(x^2 + 4y^2\right)dydx$$

$$= 4\int_0^2 \left[x^2 y + \frac{4}{3}y^3\right]_0^{\frac{1}{2}\sqrt{4-x^2}} dx$$

$$= 4\int_0^2 \left[\frac{1}{2}x^2\sqrt{4-x^2} + \frac{1}{6}\left(4-x^2\right)^{\frac{3}{2}}\right]dx$$

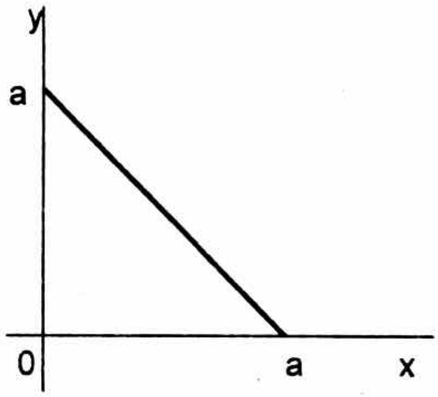
$$= 4 = \frac{4}{3}\int_0^2 \left(x^2+2\right)\sqrt{4-x^2}\,dx$$

$$= -\frac{1}{3}\left(4-x^2\right)^{\frac{3}{2}} + 2x\sqrt{4-x^2} + 8\sin^{-1}\frac{1}{2}x \;\Big]_0^2$$

$$= 4\pi$$

Physical Applications

1. A lamina in the shape of an isosceles right triangle has an area density that varies as the square of the distance from the vertex of the right angle. If the mass is measured in kg and distance is measured in m, find the mass and the center of mass of the lamina.



Solution.

The mass of the lamina is

$$M = \int_0^a \int_0^{a-x} \delta\left(x^2 + y^2\right)dydx$$

$$= \delta \int_0^a \left[yx^2 + \frac{y^3}{3} \right]_{y=0}^{y=a-x} dx$$

$$= \delta \int \left(\frac{1}{3}a^3 - a^2x + 2ax^2 - \frac{4}{3}x^3 \right)dx$$

$$= \delta\left(\frac{1}{3}a^4 - \frac{1}{2}a^4 + \frac{2}{3}a^4 - \frac{1}{3}a^4 \right)$$

$$= \frac{\delta a^4}{6}$$

To find the center of mass, the symmetry must lie on the line y = x, if we find $\bar{x}$ we will also have $\bar{y}$.

$$M_y = \int_0^a \int_0^{a-x} \delta x(x^2 + y^2)dydx$$

$$= \delta \int_0^a \left[x^3 y + \frac{xy^3}{3} \right]_{y=0}^{y=a-x} dx$$

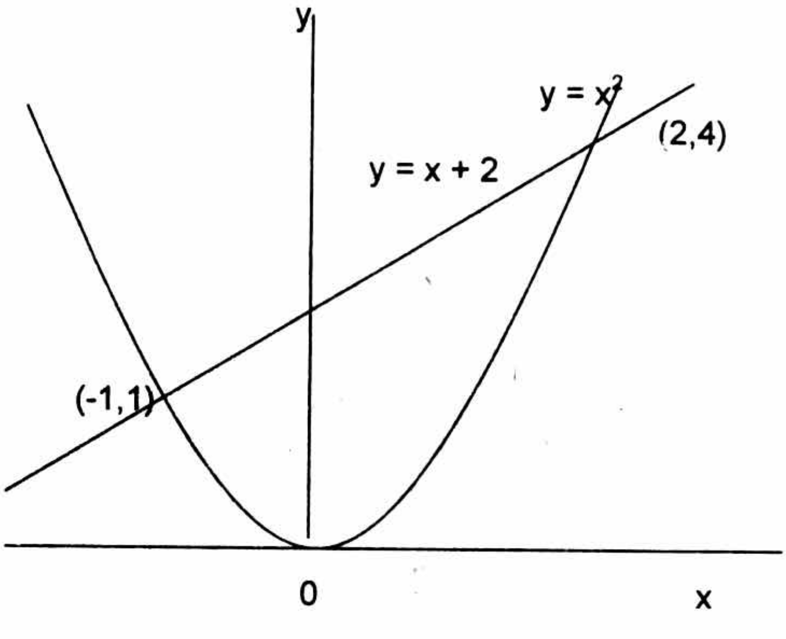
$$= \delta \int_0^a \left(\frac{a^3 x}{3} - a^2 x^2 + 2ax^3 - \frac{4x^4}{3} \right) dx$$

$$= \delta \left(\frac{a^5}{6} - \frac{a^5}{3} + \frac{a^5}{2} - \frac{4a^5}{15} \right)$$

$$= \frac{\delta a^5}{15}$$

Because $M_x = M_y$, $M_x = \frac{\delta a^5}{15}$. Thus we get $\bar{x} = \frac{2}{5}a, \bar{y} = \frac{2}{5}a$. Therefore the center of mass is at the point, $\left(\frac{2a}{5}, \frac{2a}{5} \right)$.

2. A thin plate of uniform thickness and density covers the region of the xy-plane shown in the figure



Find its moment of inertia about the y – axis.

Solution.

$$I_y = \int_{-1}^{2} \int_{x^2}^{x+2} x^2 \delta dy dx$$

Practice Exercises

Encircle the letter corresponding the correct answer. (R x 2 points)

1. $\int x \sin x \; dx$ is equal to

a. $\sin x + x \cos x + C$

b. $\sin x - x \cos x + C$

c. $-\sin x - x \operatorname{cox} x + C$

d. $-\sin x + x \cos x + C$

2. $\int \sin \alpha \; \cos \alpha \; d\alpha$ is equal to

I. $\frac{1}{2} \sin^2 \alpha + C$

II. $-\frac{1}{2} \cos^2 \alpha + C$

III. $-\frac{1}{2} \sin^2 \alpha + C$

IV. $\frac{1}{2} \cos^2 \alpha + C$

a. I only

b. I and IV

c. II only

d. I and II

3. $\int \sin^2 3\beta \; d\beta$ is equal to

a. $1/9 \sin^3 3\beta + C$

b. $1/9 \cos^3 3\beta + C$

c. $\frac{1}{2} \beta - 1/12 \sin 6\beta + C$

d. $\frac{1}{2} \beta - 1/6 \sin 3\beta + C$

4. $\int \ln e^{2x} \; dx$ is equal to

a. $x^2 + C$

b. $\frac{1}{2} x^2 + C$

c. $e^{2x} + C$

d. $\frac{1}{2} e^{2x} + C$

5. $\int x \coth x^2 \; dx$ is equal to

a. $1/3 \coth x^3 + C$

b. $\frac{1}{2} \ln \sinh x^2 + C$

c. $\frac{1}{2} \ln \cosh x^2 + C$

d. $\ln \sinh x^2 + C$

6. $\int 4 e^{3 \ln x} dx$ is equal to

a. $\frac{1}{4} x^4 + C$

b. $\frac{1}{4} e^{3 \ln x} + C$

c. $x^4 + C$

d. $e^{3 \ln x} + C$

7. $\int \frac{2dy}{3y-4}$ is equal to

a. $2/3 \ln(3y-4) + C$

b. $2 \ln(3y-4) + C$

c. $1/3 \ln(3y-4) + C$

d. $\frac{2}{3(3y-4)^2} + C$

8. $\int \frac{e^x\,dx}{e^x+1}$ is equal to

a. $(e^x + 1)^{-2} + C$

b. $e^x(e^x + 1)^{-2} + C$

c. $\ln(e^x + 1) + C$

d. $e^x(e^x + 1) + C$

9. $\int \frac{\tan\varphi\,dx}{1+\ln\cos\varphi}$ is equal to

a. $\frac{1}{2}\tan^2\varphi + C$

b. $\varphi + \ln\sin x + C$

c. $\ln(1 + \ln\cos\varphi) + C$

d. $-\ln(1 + \ln\cos\,) + C$

10. $\int \tan\theta\sec^2\theta\,d\theta$ is equal to

I. $\frac{1}{2}\tan^2\theta + C$

II. $\frac{1}{2}\sec^2\theta + C$

III. $\tan^2\theta + C$

IV. $\sec^2\theta + C$

a. I only

b. II only

c. I and II

d. III and IV

11. $\int x^n dx = \frac{1}{n+1}x^{n+1} + C$ except

a. n = -1

b. n = 0

c. n = 1

d. n > 1

12. $2\int \sin 4\theta \sin 2\theta d\theta$ is equivalent to

a. $4\int \cos 2\theta d\theta + 4\int \cos 6\theta d\theta$

b. $4\int \cos 6\theta d\theta - 4\int \cos 2\theta d\theta$

c. $\int \cos 2\theta d\theta - \int \cos 6\theta d\theta$

d. $\int \cos 2\theta d\theta + \int \cos 6\theta d\theta$

13. $3\int \cos 6y \cos 3y dy$ is equal to

a. $\frac{1}{6}\sin 9y - \frac{1}{2}\sin 3y + C$

c. $\frac{1}{3}\sin 9y - \frac{3}{2}\sin 3y + C$

b. $\frac{1}{6}\sin 9y + \frac{1}{2}\sin 3y + C$

d. $\frac{1}{3}\sin 9y + \frac{3}{2}\sin 3y + C$

14. $\int \frac{1}{2}\left(e^{2x} + e^{-2x}\right)dx$ is equal to

a. $\frac{1}{2}\left(e^{2x} - e^{-2x}\right) + C$

c. $\frac{1}{4}\left(e^{2x} + e^{-2x}\right) + C$

c. $\frac{1}{2}\left(e^{2x} + e^{-2x}\right) + C$

d. $\frac{1}{4}\left(e^{2x} - e^{-2x}\right) + C$

15. $\int_0^4 \left(x^2 - \sqrt{x}\right)dx$ is equal to

a. $\frac{64}{3}$

b. $\frac{16}{3}$

c. 8

d. 16

16. $\int_0^2 \frac{dx}{e^x}$ is equal to

a. $\ln\left|e^2 - 1\right|$

b. $e^{-2} - 1$

c. $1 - e^{-2}$

d. $1 - e^2$

17. $\int 2x\sqrt{x^2 + 1}dx$ is equal to

a. $\frac{2}{3}\left(x^2 + 1\right)^{3/2} + C$

b. $\frac{4}{3}\left(x^2 + 1\right)^{3/2} + C$

c. $\frac{3}{2}\left(x^2 + 1\right)^{3/2} + C$

d. $6\left(x^2 + 1\right)^{3/2} + C$

18. $\int_2^3 \frac{dx}{(x-1)^2}$ is equal to

a. ln 4 b. $\frac{1}{2}$

c. 0 d. $-\frac{1}{2}$

19. $\int \frac{x^2+2}{x-2} x$ is equal to

a. $\frac{x^2}{2}+2x+6\ln|x-2|+C$ c. $\frac{x^2}{2}+2x-6\ln|x-2|+C$

b. $\frac{x^2}{2}+2x+\frac{6}{x-2}+C$ d. $\frac{x^2}{2}+2x-\frac{6}{x-2}+C$

20. $\int x(x+1)^2\,dx$ is equivalent to

a. $\frac{x(x+1)^3}{3}-\frac{1}{3}\int (x+1)^3\,dx$ c. $\frac{x(x+1)^3}{3}-\frac{1}{3}\int (x+1)^2\,dx$

b. $\frac{x(x+1)^2}{2}-\frac{1}{3}\int (x+1)^3\,dx$ d. $\frac{x(x+1)^2}{2}-\frac{1}{3}\int (x+1)^2\,dx$

21. $\int xe^{-2x}\,dx$ is equivalent to

a. $\frac{xe^{-2x}}{2}-\frac{1}{4}e^{-2x}+C$ c. $\frac{xe^{-2x}}{2}+\frac{1}{4}e^{-2x}+C$

b. $-\frac{xe^{-2x}}{2}+\frac{1}{4}e^{-2x}+C$ d. $-\frac{xe^{-2x}}{2}-\frac{1}{4}e^{-2x}+C$

22.In integrating $\int x^3 \ln x\,dx$ by parts, v is

a. x^3 b. x^4

c. $\frac{x^3}{3}$ d. $\frac{x^4}{4}$

23. $\int \sin 5x \cos 3x dx$ is equal to

a. $\frac{1}{2}\left(\frac{\cos 2x}{2} - \frac{\cos 8x}{8}\right) + C$

b. $\frac{1}{2}\left(\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + C$

c. $-\frac{1}{2}\left(\frac{\cos 8x}{8} + \frac{\cos 2x}{2}\right) + C$

d. $\frac{1}{2}\left(\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right) + C$

24. $\int (\tan^2 x + \cot^2 x + 4) dx$ is equivalent to

a. $\sec^2 x + \csc^2 x + 4x + C$

b. tan x + cot x + 2x + C

c. $\sec^2 x - \csc^2 x + 4x + C$

d. tan x – cot x + 2x + C

25. $\int \frac{\sin \sqrt{x}}{\sqrt{x}} dx$ is equivalent to

a. $-\frac{1}{2}\cos\sqrt{x} + C$

b. $\frac{1}{2}\cos\sqrt{x} + C$

c. $-2\cos\sqrt{x} + C$

d. $2\cos\sqrt{x} + C$

26. $\int \tan x \sec^2 x dx$ is equal to

I) $\frac{1}{2}\tan^2 x + C$

II) $\frac{1}{2}\sec^2 x + C$

III) $\frac{1}{2}\sec^2 x \tan x + C$

a. I only

b. II only

c. I and II

d. I and III

27. $\int_0^2 2x^2\sqrt{x^3+1}dx$ is equal to

a. $\frac{104}{9}$

b. $\frac{26}{9}$

c. 27

d. 16

28. $\int_{-1}^{1} (x^{4/3} + x^{1/3}) dx$ is equal to

a. $\frac{3}{7}$

b. $\frac{6}{7}$

d. $\frac{7}{3}$

d. $\frac{7}{6}$

29. $\int_0^{\pi/2} \sin^3 \cos x dx$ is equal to

a. $\frac{1}{2}$

b. $\frac{1}{4}$

c. 2

d. 0

30. $\int_0^2 \frac{e^x + e^{-x}}{2} dx$ is approximately

a. 2.45

b. 3.63

c. 4.87

d. 5.05